Feynman-Kac techniques in functional analysis

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Question: link between $\mathbf{L}$ and $\mathbf{L} - V$ for some V?

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Analogous for the solutions to the Schrödinger equation $\partial_t u = \mathbf{L}u + b'u$ (Feynman-Kac formula):

$$u(t,x) = \mathbf{E}\left[u_0(X_t)e^{\int_0^t b'(X_s)ds} \,|\, X_0 = x\right],$$

where $(X_t)_{t\geq 0}$ solves

$$dX_t = \sqrt{2}dB_t + b(X_t)dt.$$

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- Understand the link with Schrödinger operators
  $\rightarrow$ Generalized intertwining relations, Riesz transforms
- Provide applications
  $\rightarrow$ Logarithmic Sobolev inequalities, spectral analysis

Outline

1. Intertwining relations
2. Logarithmic Sobolev inequality
3. Spectral estimates

Consider Boltzmann probability measure:

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and the diffusion process $(X_t)_{t \geq 0}$ solution of:

$$dX_t = \sqrt{2} dB_t - \nabla V(X_t) dt,$$

with $(B_t)_{t \geq 0}$ standard Brownian motion.

Main assumption:

$$CD(\rho, \infty) \qquad \boxed{\exists \rho \in \mathbf{R} : \; \nabla^2 V \geq \rho I_d.}$$

Euclidean version of the Curvature-Dimension criterion (Bakry-Émery '85)

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- if $\rho > 0$, μ satisfies stronger properties (log-Sobolev, Gaussian concentration inequalities, hypercontractivity, etc.)

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- Intertwining relation: $\nabla \mathbf{P}_t f = e^{-t} \mathbf{P}_t(\nabla f)$.

We consider vector-valued Schrödinger operators:

Definition

Let $M : \mathbf{R}^d \to \mathcal{M}_d(\mathbf{R})$ be continuous, $\mathcal{L}$ the tensor operator $\mathbf{L}^{\otimes d}$. The Schrödinger operator associated to $\mathcal{L}$ and M is $\mathcal{L}^M = \mathcal{L} - M$.

The flow of $\partial_t u = \mathcal{L}^M u$ is a Feynman-Kac semigroup denoted $(\mathcal{P}_t^M)_{t \geq 0}$.

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The flow of $\partial_t u = \mathcal{L}^M u$ is a Feynman-Kac semigroup denoted $(\mathcal{P}_t^M)_{t \geq 0}$.

- $\mathcal{L}$ acts on $F = (F_1, \ldots, F_d)$ as $\mathcal{L}F = (\mathbf{L}F_1, \ldots, \mathbf{L}F_d)$,
- M acts as the standard matrix-vector product.

Proposition (Intertwining relation)

There is an intertwining relation between $\mathbf{L}$ *and* $\mathcal{L}^{\nabla^2 V}$*:*

$$\nabla \mathbf{L} = \mathcal{L}^{\nabla^2 V} \nabla \quad (= [\mathcal{L} - \nabla^2 V] \nabla),$$

and for semigroups:

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Example

Gaussian case: $\nabla^2 V = I_d$,

$$\nabla \mathbf{P}_t f = \mathcal{P}_t^{I_d}(\nabla f) = e^{-t}\mathcal{P}_t(\nabla f).$$

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For $f : \mathbf{R}^d \to \mathbf{R}$ smooth enough, define

$$\mathrm{Ent}_\mu(f^2) = \int_{\mathbf{R}^d} f^2 \log(f^2)\, d\mu - \int_{\mathbf{R}^d} f^2\, d\mu \log\left(\int_{\mathbf{R}^d} f^2\, d\mu\right)$$

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Definition (Gross '75)

The measure μ satisfies a Logarithmic Sobolev Inequality with constant $c > 0$ (in short $LSI(c)$), if for any $f : \mathbf{R}^d \to \mathbf{R}$ smooth enough,

$$\mathrm{Ent}_\mu(f^2) \leq c \int_{\mathbf{R}^d} |\nabla f|^2 d\mu.$$

We let $c_{LSI}(\mu)$ be the optimal constant c such that μ satisfies $LSI(c)$.

- Exponential entropy decay: the measure μ satisfies $LSI(c)$, $c > 0$, if and only if, for any non-negative function f,

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- (Otto-Villani '99) Let $\nu \in \mathcal{P}_2(\mathbf{R}^d)$ such that $\nu \ll \mu$. Assume that μ satisfies $LSI(c)$, $c > 0$. Then

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- (Herbst) Gaussian concentration inequalities.
- (Gross '75) Hypercontractivity for $(\mathbf{P}_t)_{t\geq 0}$.
- etc.

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- V uniformly convex at infinity: Holley-Stroock perturbation technique. For $U : \mathbf{R}^d \to \mathbf{R}$ uniformly convex and ϕ a bounded perturbation such that $V = U + \phi$,

$$c_{LSI}(\mu) \leq \frac{2e^{4\|\phi\|_\infty}}{\inf\limits_{x\in\mathbf{R}^d} \rho_-(\nabla^2 U(x))}.$$

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- Tensorization property:

$$c_{LSI}(\mu^{\otimes n}) = c_{LSI}(\mu), \quad n \geq 1.$$

Theorem (S. '20)

Let $a : \mathbf{R}^d \to \mathbf{R}$ be a positive smooth bounded function, set

$$\kappa_a = \inf_{x \in \mathbf{R}^d} \{2\rho_-(\nabla^2 V(x)) - a\mathbf{L}(1/a)(x)\}.$$

If $\kappa_a > 0$, then

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Example

For $V(x) = |x|^4/4$, a rough optimization procedures leads to $c_{LSI}(\mu) = O(1)$ (explicit).

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Theorem (Probabilistic representation, Airault '76)

Let $f : \mathbf{R}^d \to \mathbf{R}$ *be smooth. Then*

$$\mathcal{P}_t^{\nabla^2 V}(\nabla f) = \mathbf{E}[J_t \nabla f(X_t)], \quad t \geq 0,$$

where $(J_t)_{t\geq 0}$ *solves*

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$\rightarrow$ $(J_t)_{t\geq 0}$ is the tangent flow to $(X_t)_{t\geq 0}$ w.r.t initial condition.

Second ingredient: smooth perturbation function $a : \mathbf{R}^d \to (0, +\infty)$. Let $V_a := V + \log(a^2)$ and define $d\mu_a = d\mu / a^2$, $(X_{t,a})_{t \geq 0}$, $(\mathbf{P}_{t,a})_{t \geq 0}$ and $\mathbf{L}_a$ the related stochastic objects. Define

$$R_{t,a} = \frac{a(X_{t,a})}{a(x)} \exp\left(- \int_0^t \frac{\mathbf{L}_a(a)}{a}(X_{s,a}) \, ds \right).$$

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Proposition (Girsanov representation)

Under boundedness assumptions on a, for $F : \mathcal{C}^0(\mathbf{R}^d) \to \mathbf{R}$ smooth functional,

$$\mathbf{E}[F(X_{[0,t]})|X_0 = x] = \mathbf{E}[R_{t,a}F(X_{[0,t],a})|X_{0,a} = x], \quad x \in \mathbf{R}^d, t \geq 0.$$

- Entropy: space-time integral + intertwining :

$$\mathrm{Ent}_\mu(f) = \int_{\mathbf{R}^d} \int_0^{+\infty} \frac{|\nabla \mathbf{P}_t f|^2}{\mathbf{P}_t f} \, dt d\mu$$

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- Girsanov representation + Cauchy-Schwarz :

$$\mathrm{Ent}_\mu(f)$$
$$\leq 4 \int_{\mathbf{R}^d} \int_0^{+\infty} \mathbf{E}\left[\left(\nabla\sqrt{f}^T J_t^{X_a} R_{t,a} (J_t^{X_a})^T \nabla \sqrt{f} \right) (X_{t,a}) \right] \, dt d\mu.$$

- Eigenvalue estimates on $J_t^{X_a} R_{t,a} (J_t^{X_a})^T$ and assumption on κ_a:

$$\mathrm{Ent}_\mu(f) \leq 4 \int_0^{+\infty} e^{-\kappa_a t} \int_{\mathbf{R}^d} a \mathbf{P}_{t,a}\left(a |\nabla \sqrt{f}|^2 \right) d\mu_a dt;$$

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- $\dfrac{d\mu_a}{d\mu} = 1/a^2$, boundedness $1/a$:

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Gaussian case (in $\mathbf{R}^d$): everything is known:

- $\sigma(-\mathbf{L}_\gamma) = \mathbf{N}$,
- eigenfunctions $\rightarrow$ Hermite polynomials,
- for $n \in \mathbf{N}$, $mult(n) = \binom{d-1+n}{n}$.

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Theorem (Milman '18)

If $\nabla^2 V \geq \rho I_d$, $\rho > 0$, *then*

$$\forall k \in \mathbf{N}, \qquad \lambda_k(-\mathbf{L}) \geq \rho \lambda_k(-\mathbf{L}_\gamma).$$

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Idea: correspondence between spectra of $-\mathbf{L}$ and $-\mathcal{L} + \nabla^2 V$.

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- $-\mathbf{L}$ has discrete spectrum ($\subset \mathbf{R}_+$),
- divergence-gradient decomposition:

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  ∇^* adjoint of ∇ in $L^2(\mu)$,
- Courant-Fischer characterization:

$$\lambda_k(-\mathbf{L}) = \inf_{f \perp \mathrm{span}(\mathcal{B}_k)} \frac{\int_{\mathbf{R}^d} f(-\mathbf{L}) f d\mu}{\int_{\mathbf{R}^d} f^2 d\mu},$$

  where $\mathcal{B}_k$ is the set of $k-1$ first eigenfunctions.

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Definition (Riesz transform)

We define the first Riesz transform as:

$$\mathcal{U}_0 = \nabla(-\mathbf{L})^{-1/2}.$$

- $\mathcal{U}_0$ is a unitary map,
- $\mathcal{U}_0$ and $\mathbf{L}$ satisfy the intertwining relation:

$$\mathcal{U}_0(-\mathbf{L}) = (-\mathcal{L} + \nabla^2 V)\mathcal{U}_0.$$

Up to restrictions, yields

$$-\mathbf{L} = \mathcal{U}_0^*(-\mathcal{L} + \nabla^2 V)\mathcal{U}_0.$$

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Proposition (Johnsen '01)

Provided that $\nabla^2 V \geq \rho I_d$, $\rho > 0$,

$$\sigma(-\mathbf{L})\backslash\{0\} = \sigma\left((-\mathcal{L} + \nabla^2 V)_{|R(\nabla)}\right).$$

We can get a bound on the spectral gap: for $f \in L^2(\mu)$ with $\int_{\mathbf{R}^d} f \, d\mu = 0$:

- $\mathcal{U}_0$ is an isometry:

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- $-\mathcal{L}$ is non-negative and $\nabla^2 V \geq \rho$:

$$\int_{\mathbf{R}^d} \mathcal{U}_0 f (-\mathcal{L}) \mathcal{U}_0 f d\mu + \int_{\mathbf{R}^d} \mathcal{U}_0 f \nabla^2 V \, \mathcal{U}_0 f d\mu,$$

$$\geq 0 + \rho \int_{\mathbf{R}^d} |\mathcal{U}_0 f|^2 d\mu$$

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- taking inf over f: $\lambda_1(-\mathbf{L}) \geq \rho$.

Proposition (Spectral gap)

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The technique extends to higher eigenvalues:

Theorem (Bonnefont, Joulin, S. '21)

There is an increasing sequence $(a_n)_{n\geq 0} \subset \mathbf{N}$ *s.t.*

$$\forall n \in \mathbf{N},\ \forall a_n < k \leq a_{n+1}, \quad \lambda_k(-\mathbf{L}) \geq (n+1)\rho.$$

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Proof: iteration of Riesz transforms,
Sequence: related to the kernel of Riesz transforms.

Perspectives

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- Functional inequalities:
  - covariance, concentration inequalities;
  - investigate dimensional behaviour;
  - connexions with Stein's method;
- Spectral estimates:
  - estimates on the sequence $(a_n)_{n\geq 0}$.
  - numerical aspects for spectral gap (in connection with Global Sensitivity Analysis);

- Everything:
  - D. Bakry, I. Gentil, M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, 2014;
- Markov generators, intertwining relations:
  - D. Bakry, *Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*, 1987;
  - M. Arnaudon, M; Bonnefont, A. Joulin, *Intertwinings and generalized Brascamp-Lieb inequalities*, 2018;
- Logarithmic Sobolev Inequalities:
  - F.-Y. Wang, *Modified Curvatures on Manifolds with Boundary and Applications*, 2014;
  - C. Steiner, *A Feynman-Kac approach for Logarithmic Sobolev Inequalities*, 2021;
- Spectral estimates:
  - E. Milman, *Spectral Estimates, Contractions and Hypercontractivity*, 2018;
  - M. Bonnefont, A. Joulin, *Intertwinings, second-order Brascamp–Lieb inequalities and spectral estimates*, 2021;

Any questions?