Journées de probabilités 2021

Stochastic resonance in stochastic PDEs

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A sample model of Stochastic Resonance

Perturbations:

- ▷ A deterministic periodic driving force
- ▷ An additive noise

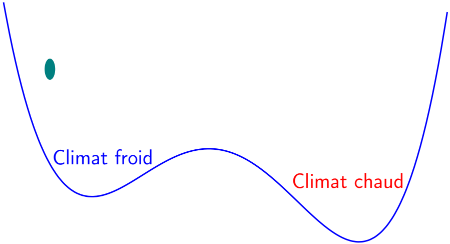

Figure: A double-well potential.

Stochastic PDE

$$\mathrm{d}\,\phi(t,x) = \frac{1}{\varepsilon}[\Delta\phi(t,x) + f(t,\phi(t,x))]\,\mathrm{d}\,t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}\,W(t,x)$$

- $\phi(t,x) \in \mathbb{R}$, $t \geqslant 0$, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}, L > 0$
- $0 \leqslant \varepsilon, \sigma \ll 1$
- $f : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that $f(t,\phi) = -\partial_\phi U(t,\phi)$
- $\mathrm{d}W(t,x) = \xi(t,x)\,\mathrm{d}t$ where
  - ξ space-time white noise: centered, Gaussian, $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$
  - ξ distribution, $\langle\xi,\varphi\rangle \sim \mathcal{N}(0,\|\varphi\|^2_{L^2})$, $\mathbb{E}[\langle\xi,\varphi_1\rangle\langle\xi,\varphi_2\rangle] = \langle\varphi_1,\varphi_2\rangle$

Stochastic Allen-Cahn PDE

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3 + A\cos(t)\Big]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\mathrm{d}W(t,x)$$

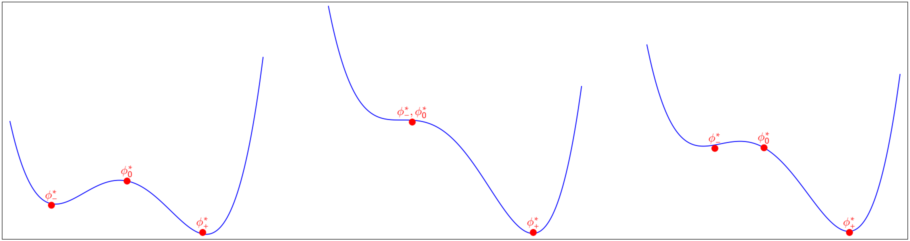

Figure: The potential $U(t,\phi) = \frac{1}{4}\phi^4 - \frac{1}{2}\phi^2 - A\cos(t)\phi$, from which the drift term is derived ($t = 0.5$). For $A \ll A_c$, the potential is asymmetric (left). For $A = A_c$, the left-hand well and the saddle meet(middle). For $A < A_c$, the left-hand well approaches the saddle (right).

Stochastic Resonance

Stationary solutions:

- If $A = 0$, $\phi(t,x) - \phi(t,x)^3 = 0$ has exactly three solutions: ± 1 and 0
- If $A \neq 0$ and $A < A_c = \frac{2}{3\sqrt{3}}$, $\phi(t,x) - \phi(t,x)^3 + A\cos(t) = 0$ has exactly three solutions: $\phi^*_-(t) < \phi^*_0(t) < \phi^*_+(t)$

Added noise: Mechanism of Stochastic Resonance

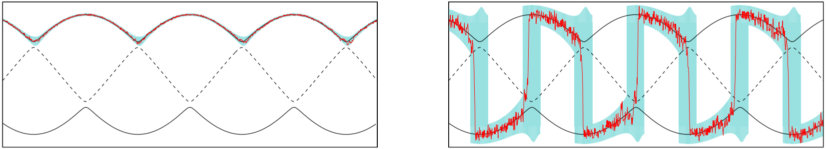

Figure: Sample paths of the SDE $\mathrm{d}\phi_t = \frac{1}{\varepsilon}\left[\phi_t - \phi_t^3 + A\cos(t)\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\mathrm{d}W_t$ for $\varepsilon = 0.005$, and $\sigma = 0.02$ (left picture) and $\sigma = 0.14$ (right picture).

Stable case: Deterministic dynamics

- Define the fractional Sobolev norm of ϕ by $\|\phi\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}\phi_k^2$
- Assume that

$$\exists \phi^* : I \to \mathbb{R};\, f(t,\phi^*(t)) = 0 \text{ and } \partial_\phi f(t,\phi^*(t)) < 0 \quad \forall t \in I = [0,T]$$

Proposition 1:

$\exists\ C, \varepsilon_0 > 0$; for $0 < \varepsilon < \varepsilon_0$,

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\left[\Delta\phi(t,x) + f(t,\phi(t,x))\right]\mathrm{d}t$$

admits a solution $\bar{\phi}(t,x)$ satisfying

$$\|\bar{\phi}(t,x) - \phi^*(t)e_0\|_{H^1} \leqslant C\varepsilon \qquad \forall t \in [0,T],$$

where $e_0(x) = \frac{1}{\sqrt{L}}$.

Stable case: Stochastic dynamics

Define for any $h > 0$,

- $\mathcal{B}(h) = \Big\{(t,\phi)\colon t \in I, \|\phi - \bar{\phi}(t,\cdot)\|_{H^s} < h\Big\}$
- the first-exit time from $\mathcal{B}(h)$: $\tau_{\mathcal{B}(h)} = \inf\Big\{t > 0\colon \|\phi - \bar{\phi}(t,\cdot)\|_{H^s} \geqslant h\Big\}$

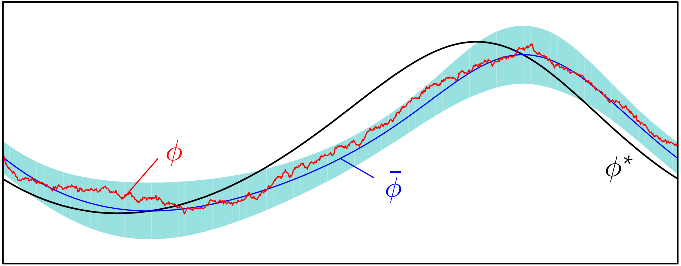

Figure: Concentration of sample paths near a stable equilibrium branch $\phi^*(t)$.

Stable case: Stochastic dynamics

Theorem 1:

For any $s \in (0, \frac{1}{2})$ and $\nu > 0$, $\exists\ \varepsilon_0, h_0, \bar\kappa, C(t,\varepsilon,s) > 0$; whenever $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < h \leqslant h_0\varepsilon^{\nu}$ one has

$$\mathbb{P}\Bigl\{\tau_{\mathcal{B}(h)} < t\Bigr\} \leqslant C(t,\varepsilon,s)\exp\biggl\{-\bar\kappa\frac{h^2}{2\sigma^2}\biggl[1-\mathcal{O}\biggl(\frac{h}{\varepsilon^{\nu}}\biggr)\biggr]\biggr\}\,.$$

Stable case: Stochastic dynamics

Theorem 1:

For any $s \in (0, \frac{1}{2})$ and $\nu > 0$, $\exists\ \varepsilon_0, h_0, \bar{\kappa}, C(t, \varepsilon, s) > 0$; whenever $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < h \leqslant h_0 \varepsilon^\nu$ one has

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- Equation for the deviation $\psi(t, x) = \phi(t, x) - \overline{\phi}(t, x)$:

$$\mathrm{d}\psi(t, x) = \frac{1}{\varepsilon}[\Delta\psi(t, x) + \bar{a}(t)\psi(t, x) + b(t, \psi(t, x))]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t, x).$$

- A solution

$$\psi(t, \cdot) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\, \mathrm{e}^{[(t-s)/\varepsilon]\Delta}\, \mathrm{d}W(s, \cdot)$$
$$+ \frac{1}{\varepsilon} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\, \mathrm{e}^{[(t-s)/\varepsilon]\Delta}\, b(s, \psi(s, \cdot))\,\mathrm{d}s = \psi^0(t, \cdot) + \psi^1(t, \cdot)$$

Stable case: Stochastic dynamics

- $\mathbb{P}\Big\{\sup_{t\in I}\|\psi^0(t,\cdot)\|_{H^s} > h_0\Big\} \leqslant C(t,\varepsilon,s)\,\mathrm{e}^{-\bar\kappa h_0^2/2\sigma^2}$
  - $\mathbb{P}\Big\{\sup_{t\in I}|\psi_k(t)| > h\Big\} \leqslant C_k(t,\varepsilon)\exp\big\{-\kappa\langle k\rangle^2\frac{h^2}{\sigma^2}\big\}$, $C_k(t,\varepsilon) = \frac{2c_0^-\langle k\rangle^2}{\gamma\varepsilon}T$
  - $h_0^2 = \sum_{k\in\mathbb{Z}} h_k^2$,
    $\mathbb{P}\Big\{\sup_{t\in I}\|\psi^0(t,\cdot)\|_{H^s} > h_0\Big\} \leqslant \sum_{k\in\mathbb{Z}}\mathbb{P}\bigg\{\sup_{t\in I}|\psi_k(t)|^2 > h_k^2\langle k\rangle^{-2s}\bigg\}$
  - Choosing $h_k^2 = C(\eta,s)h^2\langle k\rangle^{-2+2s+\eta}$, $0<\eta<2\rho$
  - $\sum_{k\in\mathbb{Z}}\langle k\rangle^2\,\mathrm{e}^{-\beta\langle k\rangle^\eta} < \infty$

Stable case: Stochastic dynamics

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  - $\mathbb{P}\Big\{\sup_{t\in I}|\psi_k(t)| > h\Big\} \leqslant C_k(t,\varepsilon)\exp\{-\kappa\langle k\rangle^2\frac{h^2}{\sigma^2}\}$, $C_k(t,\varepsilon) = \frac{2c_0^-\langle k\rangle^2}{\gamma\varepsilon}T$
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  - Choosing $h_k^2 = C(\eta,s)h^2\langle k\rangle^{-2+2s+\eta}$, $0<\eta<2\rho$
  - $\sum_{k\in\mathbb{Z}}\langle k\rangle^2\,\mathrm{e}^{-\beta\langle k\rangle^\eta} < \infty$

- $\mathbb{P}\Big\{\sup_{0\leqslant t\leqslant T\wedge\tau_{\mathcal{B}(h)}}\|\psi^1(t,\cdot)\|_{H^s} > h_1, \sup_{0\leqslant t\leqslant T}\|\psi^0(t,\cdot)\|_{H^s} \leqslant h_0\Big\} = 0$
  - Assume $\psi(t,\cdot)\in H^s$ for all $0<s<\frac{1}{2}$ then $\beta(t)\in H^r$ for all $r<\frac{1}{2}$
  - $\exists M'(q,r)<\infty$; $\forall q<r+2$, $\psi^1(t,\cdot)\in H^q$ and

$$\|\psi^1(t,\cdot)\|_{H^q} \leqslant M'(q,r)\varepsilon^{\frac{q-r}{2}-1}\sup_{0\leqslant s\leqslant t}\|\beta(s)\|_{H^r}\,.$$

  - Choosing $h_1 = M'(q,r)\varepsilon^{\frac{q-r}{2}-1}Mh^2$.

Unstable case

In a neighbourhood of $(0,0)$,

$$d\phi(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi(t,x) + g(t) - \phi(t,x)^2 - b(t,\phi(t,x))\Big]dt + \frac{\sigma}{\sqrt{\varepsilon}}dW(t,x) \tag{1}$$

$$\Updownarrow \phi(t,x) = \phi_0(t)e_0(x) + \phi_\perp(t,x)$$

Coupled SDE-SPDE system:

$$d\phi_0 = \frac{1}{\varepsilon}\Bigg[g(t) - \phi_0^2 - b(t,\phi_0(t)e_0) + b_0(t,\phi_0(t),\phi_\perp(t))\Bigg]dt + \frac{\sigma}{\sqrt{\varepsilon}}dW_0$$

$$d\phi_\perp = \frac{1}{\varepsilon}\Bigg[\Delta\phi_\perp + a(t,\phi_0)\phi_\perp + b_\perp(t,\phi_0(t),\phi_\perp(t))\Bigg]dt + \frac{\sigma}{\sqrt{\varepsilon}}dW_\perp$$

- $g(t) = \delta + t^2 + \mathcal{O}(t^3), \delta \geqslant 0$ and $b(t,\phi) = \mathcal{O}(\phi^3)$
- $\phi_\pm^*(t)$ such that $f(t,\phi_\pm^*(t)) = 0$, $\partial_\phi f(t,\phi_\pm^*(t)) \asymp \mp\big(\sqrt{\delta} + |t|\big)$
- $a(t,\phi_0) = \mathcal{O}(-2\phi_0(t)) < 0$ and $b_0 = b_\perp = \mathcal{O}(\|\phi_\perp\|_{H^1}^2)$.

Deterministic dynamics near the origin

Proposition 2:

Equation given by (1) with $\sigma = 0$ admits $\phi_\perp(t,x) = 0$ and ϕ_0 obeys

$$\varepsilon\dot{\phi}_0(t) = g(t) - \phi_0(t)^2 - b(t,\phi_0(t)e_0).$$

[N.B. & Barbara Gentz 2002]:
$\exists\ \bar{\phi}_0(t)$ tracking $\phi_+^\star(t)$:

$$\exists\, T_0, c_0 > 0;\ \bar{\phi}_0(t) - \phi_+^*(t) \asymp \begin{cases} \dfrac{\varepsilon}{|t|} & \text{for } -T_0 \leqslant t \leqslant -c_0 \max\{\sqrt{\delta},\sqrt{\varepsilon}\} \\ -\dfrac{\varepsilon}{|t|} & \text{for } c_0 \max\{\sqrt{\delta},\sqrt{\varepsilon}\} \leqslant t \leqslant T_0. \end{cases}$$

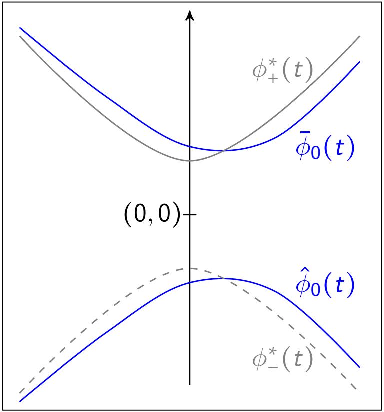

Figure: Equilibrium branches and associated adiabatic solutions near $(0,0)$.

Transverse stochastic dynamics for $\phi_\perp$

Given given $h_\perp > 0$, we define the set

$$\mathcal{B}_\perp(h_\perp) = \Big\{(t,\phi): t \in [-T_0, T_0], \|\phi_\perp\|_{H^s} < h_\perp\Big\} .$$

Behaviour of $\phi_\perp(t,x)$:

Theorem 2:

For any $s \in (0, \frac{1}{2})$ and $\nu > 0$, $\exists \varepsilon_0, h_0, \bar{\kappa}, C(t,\varepsilon,s) > 0$; whenever $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < h_\perp \leqslant h_{0\perp}\varepsilon^\nu$ one has

$$\mathbb{P}\Big\{\tau_{\mathcal{B}_\perp(h_\perp)} < t\Big\} \leqslant C(t,\varepsilon,s)\exp\left\{-\bar{\kappa}\frac{h_\perp^2}{2\sigma^2}\left[1 - \mathcal{O}\left(\frac{h_\perp}{\varepsilon^\nu}\right)\right]\right\} .$$

Stochastic dynamics near $\bar\phi_0(t)$

- $\zeta(t) \asymp \frac{1}{|\bar a(t,\bar\phi_0)|}$ and $\hat\zeta(t) = \sup_{-T_0 \leqslant s \leqslant t} \zeta(s) \quad \forall t \in [-T_0, T_0]$
- $\mathcal{B}_0(h) = \Big\{(t,\phi_0) : t \in [-T_0, T_0], |\phi_0 - \bar\phi_0(t)| < h\sqrt{\zeta(t)}\Big\}$

Theorem 3:

$\exists\ \varepsilon_0, h_0, \kappa, C(t,\varepsilon) > 0$; whenever $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < h \leqslant h_0 \hat\zeta(t)^{-3/2}$, one has

$$\mathbb{P}\Big\{\tau_{\mathcal{B}_0(h)} < t, \tau_{\mathcal{B}_\perp(h_\perp)} > t\Big\} \leqslant C(t,\varepsilon)\exp\bigg\{-\kappa\frac{h^2}{2\sigma^2}\bigg\},$$

where $\kappa = 1 - \mathcal{O}(h\hat\zeta(t)^{3/2})$.

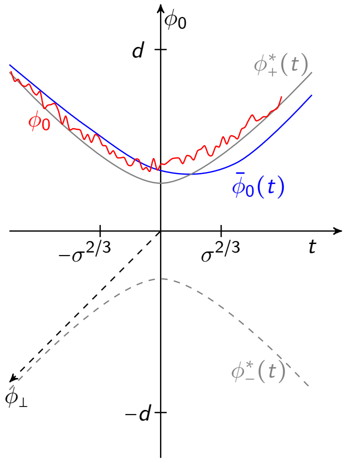

Figure: Weak noise regime

Noise regime

- ▷ Weak-noise regime: if $\sigma \ll \max\{\varepsilon, \delta\}^{3/4}$, Theorem 3 can be applied for any $t \in [-T_0, T_0]$, and shows that $\phi_0(t)$ remains close to $\bar{\phi}_0(t)$ with high probability during the whole time interval.

Noise regime

- Weak-noise regime: if $\sigma \ll \max\{\varepsilon, \delta\}^{3/4}$, Theorem 3 can be applied for any $t \in [-T_0, T_0]$, and shows that $\phi_0(t)$ remains close to $\bar{\phi}_0(t)$ with high probability during the whole time interval.
- Strong-noise regime: if $\sigma \geqslant \max\{\varepsilon, \delta\}^{3/4}$, Theorem 3 can only be applied up to times t of order $-\sigma^{2/3}$, showing that $\phi_0(t)$ is unlikely to become negative up to times of that order.

Stong-noise regime

Theorem 4:

Assume $\sigma \geqslant \max\{\varepsilon, \delta\}^{3/4}$, fix $d > 0$.
Let $h > 0$;

$$\bar{\phi}_0(t) + h\sqrt{\zeta(t)} \leqslant d \quad \forall t \in [-\sigma^{2/3}, \sigma^{2/3}].$$

$\exists\ \kappa > 0$ and $\varrho \geqslant 1$;

$$\mathbb{P}\Big\{\phi_0(s) > -d \ \forall s \in [-\sigma^{2/3}, t], \tau_{\mathcal{B}_\perp(h_\perp)} > t\Big\}$$
$$\leqslant \frac{3}{2}\exp\left\{-\kappa\frac{\sigma^{4/3}}{\varrho\varepsilon}\right\} + C(t,\varepsilon)\,\mathrm{e}^{-h^2/\sigma^2},$$

$\forall t \in [-\sigma^{2/3}, \sigma^{2/3}]$.

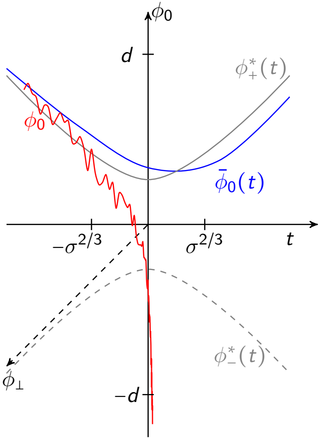

Figure: Strong noise regime.

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Thanks for listening!