# Long time behaviour for some time-inhomogeneous stochastic kinetic models with jumps

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# Introduction

dynamics of an object under a frictional force

Newton's second law: $\begin{cases} v_t' = -\frac{1}{2}\nabla U(v_t), & U \text{ potential} \\ x_t' = v_t \end{cases}$

- $\nabla U(0) = 0$
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## Frictional forces

- classical: $b(v) = k|v|,\ k > 0$
- fluid dynamics (Rayleigh) : $b(v) = kv^2,\ k > 0$
- aerodynamics : $b(v) = k|v|^3,\ k > 0$

# Langevin

Add a random perturbation force:

$$\begin{cases} \mathrm{d}V_t = -V_t\,\mathrm{d}t + \mathrm{d}B_t \\ \mathrm{d}X_t = V_t\,\mathrm{d}t \end{cases}$$

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## Persistent Turning Walker Model

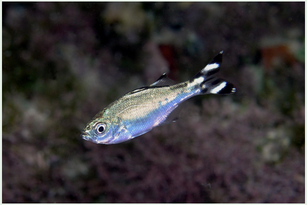

Figure: *Kuhlia Mugil*

$(\theta_t, \kappa_t)$: velocity angle and curvature

$$\begin{cases} \mathrm{d}\kappa_t = -\kappa_t\,\mathrm{d}t + \sqrt{2}\alpha\,\mathrm{d}B_t \\ \mathrm{d}\theta_t = \kappa_t\,\mathrm{d}t \end{cases}$$

# What about "kinetic" SDE ?

$$\begin{cases} V_t = V_0 + L_t - \int_0^t \frac{1}{2} \partial_v U(s, V_s)\, \mathrm{d}s, \\ X_t = X_0 + \int_0^t V_s\, \mathrm{d}s, \end{cases} \tag{1}$$

$X_t$: position at time $t$ of a particle, with speed $V_t$.
$(V_t)_{t \geq 0}$ can be seen as a random process in a potential $U(t,v))_{t \geq 0, v \in \mathbb{R}}$.

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**Existence of solution ? Explosion ? Behaviour of $(r_\epsilon X_{t/\epsilon})_{t\geq 0}$, as $\epsilon \to 0$ ?**

## Generalization of the Langevin model ?

- ⇝ Non linearity of the frictional force:
- ⇝ Time non-homogeneity of the frictional force:
- ⇝ Discontinuity of the random force ⇢ jumps

(Fournier-Tardif, 2021)

$$\begin{cases} V_t = V_0 + B_t + \dfrac{\rho}{2}\displaystyle\int_0^t \dfrac{\Theta'}{\Theta}(V_s)\,\mathrm{d}s, \;\; \rho \geq 0 \\ X_t = X_0 + \displaystyle\int_0^t V_s\,\mathrm{d}s \end{cases}$$

for some even $\Theta : \mathbb{R} \mapsto (0, \infty)$ $C^2$ satisfying $\lim_{|v|\to\infty} |v|\Theta(v) = 1$.

$$\textbf{f.d.d.-}\lim_{\epsilon\to 0}(r_\epsilon X_{t/\epsilon})_{t\geq 0}, \;\; \textbf{with } r_\epsilon \xrightarrow[\epsilon\to 0]{} 0$$

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(Gradinaru-Offret, 2013)

For $\rho, \beta \in \mathbb{R}$, $\gamma > -1$,

$$V_t = V_0 + B_t + \rho \int_0^t \frac{\operatorname{sgn}(V_s)\,|V_s|^\gamma}{s^\beta}\,\mathrm{d}s$$

$$\boldsymbol{\mathcal{L}-\lim_{t\to\infty} r_t V_t,\ \text{with } r_t \underset{t\to\infty}{\longrightarrow} 0}$$

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(Eon-Gradinaru 2015)

$$\begin{cases} V_t^\epsilon = \epsilon L_t - \int_0^t \operatorname{sgn}(V_s^\epsilon)\,|V_s^\epsilon|^\gamma\,\mathrm{d}s, \ \gamma + \frac{\alpha}{2} > 2, \\ X_t^\epsilon = X_0 + \int_0^t V_s^\epsilon\,\mathrm{d}s. \end{cases}$$

$$\boldsymbol{\mathcal{L} - \lim_{\epsilon \to 0} r_\epsilon X^\epsilon_{\frac{t}{\epsilon^\alpha}}}$$

# The model

$$\begin{cases} \mathrm{d}V_t = \mathrm{d}L_t - \dfrac{F(V_t)}{t^\beta}\,\mathrm{d}t, & V_{t_0} = v_0 > 0 \\ \mathrm{d}X_t = V_t\,\mathrm{d}t, & X_{t_0} = x_0 \in \mathbb{R}. \end{cases} \tag{SKE}$$

$(L_t)_{t\geq 0}$ is a Lévy process, $\beta \in \mathbb{R}$.
$F$ is supposed to satisfy

$$\text{for some } \gamma \in \mathbb{R},\ \forall v \in \mathbb{R},\ \lambda > 0,\ F(\lambda v) = \lambda^\gamma F(v). \tag{$H^\gamma$}$$

When $\gamma \geq 1$, assume for all $v \in \mathbb{R}$, $vF(v) \geq 0$.

**Keep in mind**

$$\forall t \geq t_0,\ V_t = v_0 - L_{t_0} + L_t - \int_{t_0}^{t} \rho\,\mathrm{sgn}(V_s)\frac{|V_s|^\gamma}{s^\beta}\,\mathrm{d}s.$$

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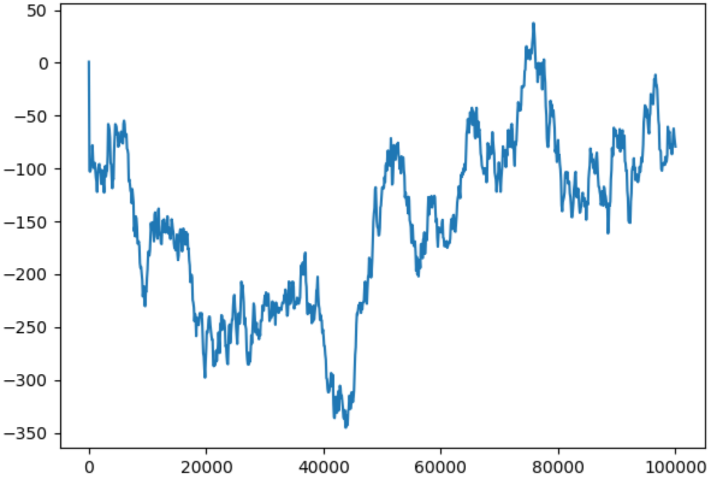
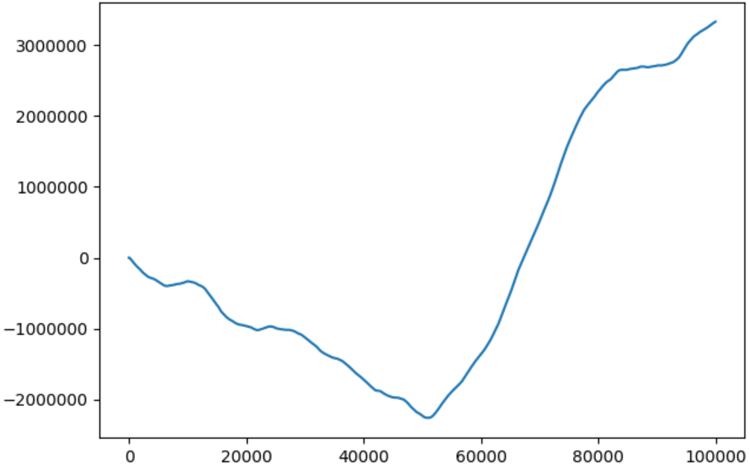
Path of the velocity process on [1, 100000] with drift $F(v) = sgn(v)|v|^{\gamma}$, $(\gamma,\beta)=(1, 2)$



Path of the position process on [1, 100000] with drift $F(v) = sgn(v)|v|^{\gamma}$, $(\gamma,\beta)=(1, 2)$



$$r_{\epsilon,X} X_{t/\epsilon} = r_{\epsilon,X} x_0 + \int_\epsilon^t \underbrace{\frac{r_{\epsilon,X}}{\epsilon}}_{:=r_{\epsilon,V}} V_{s/\epsilon}\,\mathrm{d}s.$$

$$r_{\epsilon,V} V_{t/\epsilon} = r_{\epsilon,V}(v_0 - B_1) + r_{\epsilon,V} B_{t/\epsilon} - r_{\epsilon,V}^{1-\gamma} \epsilon^{\beta-1} \int_\epsilon^t r_{\epsilon,V}^{\gamma} F\left(\frac{V_{u/\epsilon}}{r_{\epsilon,V}}\right) u^{-\beta}\,\mathrm{d}u.$$

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$\rightsquigarrow r_{\epsilon,V} = \sqrt{\epsilon}$ and $r_{\epsilon,X} = \epsilon^{3/2}$ ?

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$\rightsquigarrow$ $r_{\epsilon,V} = \sqrt{\epsilon}$ and $r_{\epsilon,X} = \epsilon^{3/2}$ ?

Call $q := \dfrac{\beta}{\gamma + 1}$

### Moment estimates (Gradinaru-L.)

Assume $q \geq \frac{1}{2}$. The inequality

$$\forall t \geq t_0, \ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{\frac{\kappa}{2}}$$

yields for

- $\kappa \in [0,1]$ when $\gamma < 1$,
- $\kappa \geq 0$, when for all $v \in \mathbb{R}$, $vF(v) \geq 0$.

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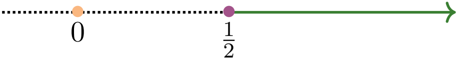


Consider $\gamma \geq 0$. Let $(\mathcal{B}_t)_{t\geq 0}$ be a standard Brownian motion. In the space of continuous functions $\mathcal{C}((0,\infty),\mathbb{R})$ endowed by the uniform topology:

**Theorem ($q = 0$ i.e. $\beta = 0$, for $F(x) = \rho \operatorname{sgn}(x)\,|x|^{\gamma}$, Eon-Gradinaru, 2015)**

As $\epsilon \to 0$,

$$(\sqrt{\epsilon} X_{t/\epsilon})_{t\geq 0} \overset{\text{f.d.d}}{\Longrightarrow} (\sigma_F \mathcal{B}_t)_{t\geq 0}.$$

**Theorem ($q > \frac{1}{2}$, Gradinaru-L.)**

As $\epsilon \to 0$,

$$(\sqrt{\epsilon} V_{t/\epsilon}, \epsilon^{3/2} X_{t/\epsilon})_{t\geq \epsilon t_0} \Longrightarrow (\mathcal{B}_t, \int_0^t \mathcal{B}_s \, \mathrm{d}s)_{t>0}.$$

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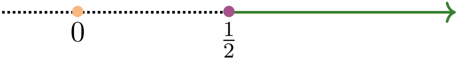


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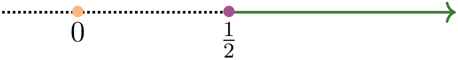



**Theorem ($q = \frac{1}{2}$, Gradinaru-L.)**

Let $\tilde{H}$ be the ergodic process ,starting at its invariant measure, solution to

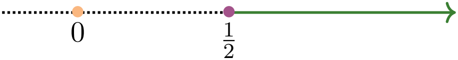
$$\mathrm{d}\tilde{H}_s = \mathrm{d}W_s - \frac{\tilde{H}_s}{2}\,\mathrm{d}s - F\big(\tilde{H}_s\big)\,\mathrm{d}s,$$

$(W_t)_{t\geq 0}$ being a standard Brownian motion. Call $\nu_{F,t_1,\cdots,t_d}$ its f.d.d.
As $\epsilon \to 0$,

$$(\sqrt{\epsilon}V_{t/\epsilon}, \epsilon^{3/2}X_{t/\epsilon})_{t\geq \epsilon t_0} \Longrightarrow (\mathcal{V}_t, \int_0^t \mathcal{V}_s\,\mathrm{d}s)_{t>0}.$$

The f.d.d. of the process $(\mathcal{V}_t)_{t\geq 0}$ are given by
$T * \nu_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$,
$T : u := (u_1, \cdots, u_d) \mapsto (\sqrt{t_1}u_1, \cdots, \sqrt{t_d}u_d)$.

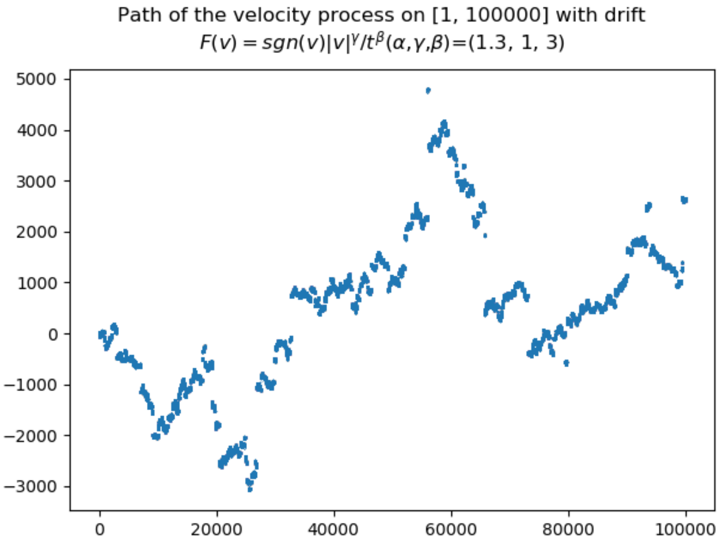
$$q = \frac{\beta}{\gamma + 1}$$

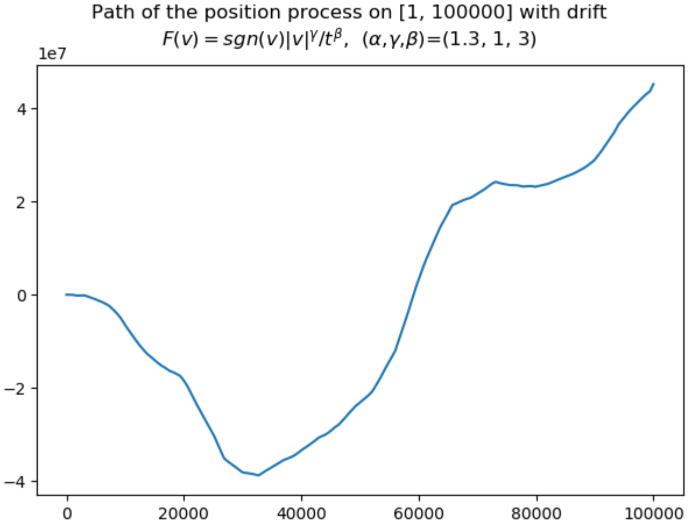



**Theorem ($q < \frac{1}{2}$, for $F(v) = \rho \operatorname{sgn}(v) |v|^\gamma$, Gradinaru-Offret, 2013)**

Call $\nu_F(\mathrm{d}x) = \frac{1}{Z} e^{-\frac{2\rho}{\gamma+1}|x|^{\gamma+1}} \mathrm{d}x$.
Then, as $\epsilon \to 0$, for all $t \geq t_0$,

$$\frac{\epsilon^q V_{t/\epsilon}}{t^q} \Longrightarrow \nu_F.$$


Path of the velocity process on [1, 100000] with drift
$F(v) = sgn(v)|v|^{\gamma}/t^{\beta} (\alpha,\gamma,\beta)=(1.3, 1, 3)$
5000
4000
3000
2000
1000
0
−1000
−2000
−3000
0
20000
40000
60000
80000
100000


Path of the position process on [1, 100000] with drift
$F(v) = sgn(v)|v|^{\gamma}/t^{\beta}$, $(\alpha,\gamma,\beta)$=(1.3, 1, 3)
1e7
4
2
0
−2
−4
0
20000
40000
60000
80000
100000

# Stable process

Fix $\alpha \in (0,2)$.

## Definition

A *(symmetric)* $\alpha$*- stable process* $(L_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t)$*-adapted process s.t*

- $L_0 = 0$ *a.s.*
- $t \mapsto L_t$ *is a.s. càdlàg*
- *for* $0 \leq s < t$*, the increment* $L_t - L_s$ *is independent of* $\mathcal{F}_s$ *and* $L_t - L_s \sim L_{t-s} \sim \mathcal{S}(\alpha, (t-s)^{1/\alpha}, 0, 0)$.

# Stable process

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## Definition

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## Scaling property

For $c > 0$, $\left(\frac{L_{ct}}{c^{1/\alpha}}\right)_{t \geq 0}$ is again an $\alpha$-stable process.

## Moment estimates (Gradinaru-L.)

Assume $\beta \geq 1 + \frac{\gamma - 1}{\alpha}$. The inequality

$$\forall t \geq t_0, \ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{p_\alpha(\gamma,\kappa)}$$

yields with $p_\alpha(\gamma,\kappa)$ defined by

| $\alpha \in (0,1)$ | | $\alpha \in (1,2)$ | | |
|---|---|---|---|---|
| | $\kappa \in [0,\alpha)$ | | $\kappa \in [0,1]$ | $\kappa \in (1,\alpha)$ |
| $\gamma \geq 0$ | $\kappa/\alpha$ | $\gamma \in [0,1)$ | $\kappa/\alpha$ | $3\kappa/2\alpha$ |
| | | $\gamma \geq 1$ | $\kappa$ | $\kappa/\alpha + \kappa/2$ |

Consider $\gamma \in [0, \alpha)$. Let $(\mathcal{S}_t)_{t \geq 0}$ be an $\alpha$-stable process.

**Theorem ($\beta > 1 + p_\alpha(\gamma) - \frac{1}{\alpha}$, Gradinaru-L.)**

Consider and $\beta \geq 0$. Define $p_\alpha(\gamma)$ by

| $\alpha \in (0,1)$ | $\alpha \in (1,2)$ |
|---|---|
| $\gamma/\alpha$ | $\begin{cases} \gamma/\alpha & \text{if } \gamma \in [0,1) \\ \gamma & \text{if } \gamma = 1 \\ \gamma\left(\frac{1}{\alpha} + \frac{1}{2}\right) & \text{if } \gamma \in (1, \alpha). \end{cases}$ |

Suppose also that for all $v \in \mathbb{R}$, $vF(v) \geq 0$ , when

- $\alpha \in (0,1)$
- or $\alpha \in (1,2)$ and $\gamma \geq 1$.

Then, as $\epsilon \to 0$, in $\mathcal{D}((0,\infty), \mathbb{R})$ (Skorokhod topology),

$$(\epsilon^{1/\alpha} V_{t/\epsilon}, \epsilon^{1+1/\alpha} X_{t/\epsilon})_{t \geq \epsilon t_0} \Longrightarrow (\mathcal{S}_t, \int_0^t \mathcal{S}_s \, \mathrm{d}s)_{t>0}, \tag{2}$$

## Theorem ($\beta = 1 + \frac{\gamma - 1}{\alpha}$, Gradinaru-L.)

Let $\tilde{H}$ be the ergodic process ,starting at its invariant measure, solution to

$$\mathrm{d}\tilde{H}_s = \mathrm{d}R_s - \frac{\tilde{H}_s}{\alpha}\,\mathrm{d}s - F\big(\tilde{H}_s\big)\,\mathrm{d}s,$$

$(R_t)_{t\geq 0}$ being an $\alpha$-stable process. Call $\nu_{F,t_1,\cdots,t_d}$ its f.d.d.
As $\epsilon \to 0$,

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$T : u := (u_1, \cdots, u_d) \mapsto (t_1^{1/\alpha} u_1, \cdots, t_d^{1/\alpha} u_d)$.

# Step 1. From convergence of $V$ to convergence of $X$

Fix $\epsilon \in (0,1]$ and $t \geq \epsilon$. Call $(V_t^{(\epsilon)})_{t>0} := (\epsilon^{1/\alpha} V_{t/\epsilon})_{t>0}$.

**If $\sup_{\epsilon \leq t \leq T} \left| V_t^{(\epsilon)} - L_t^{(\epsilon)} \right| \xrightarrow{\mathbb{P}} 0$, as $\epsilon \to 0$ is proved.**

One can write

$$X_t^{(\epsilon)} := \epsilon^{1+1/\alpha} X_{t/\epsilon} = \epsilon^{1+1/\alpha} x_0 + \int_\epsilon^t \epsilon^{1/\alpha} V_{s/\epsilon}\, \mathrm{d}s.$$

So

$$(V_\bullet^{(\epsilon)}, X_\bullet^{(\epsilon)}) = g_\epsilon(V_\bullet^{(\epsilon)}),$$

where $g_\epsilon : v \in \mathcal{D} \mapsto \left(v_t, \int_\epsilon^t v_s\, \mathrm{d}s\right)_{t>0}$ converges as $\epsilon \to 0$, to $g : v \mapsto \left(v_t, \int_0^t v_s\, \mathrm{d}s\right)_{t>0}$.

# Step 1. From convergence of $V$ to convergence of $X$

Fix $\epsilon \in (0,1]$ and $t \geq \epsilon$. Call $(V_t^{(\epsilon)})_{t>0} := (\epsilon^{1/\alpha} V_{t/\epsilon})_{t>0}$.

**If $\sup_{\epsilon \leq t \leq T} \left| V_t^{(\epsilon)} - L_t^{(\epsilon)} \right| \xrightarrow{\mathbb{P}} 0$, as $\epsilon \to 0$ is proved.**

One can write

$$X_t^{(\epsilon)} := \epsilon^{1+1/\alpha} X_{t/\epsilon} = \epsilon^{1+1/\alpha} x_0 + \int_\epsilon^t \epsilon^{1/\alpha} V_{s/\epsilon} \, \mathrm{d}s.$$

So

$$(V_\bullet^{(\epsilon)}, X_\bullet^{(\epsilon)}) = g_\epsilon(V_\bullet^{(\epsilon)}),$$

where $g_\epsilon : v \in \mathcal{D} \mapsto \left( v_t, \int_\epsilon^t v_s \, \mathrm{d}s \right)_{t>0}$ converges as $\epsilon \to 0$, to $g : v \mapsto \left( v_t, \int_0^t v_s \, \mathrm{d}s \right)_{t>0}$.

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# Step 2a. Convergence of $V$ for $\beta > 1 + p_\alpha(\gamma) - \frac{1}{\alpha}$

**Prove that $\sup_{\epsilon \le t \le T} \left| V_t^{(\epsilon)} - L_t^{(\epsilon)} \right| \xrightarrow{\mathbb{P}} 0$, as $\epsilon \to 0$.**

$$V_t^{(\epsilon)} = \epsilon^{1/\alpha}(v_0 - L_1) + L_t^{(\epsilon)} - \epsilon^{\beta - 1 + (1-\gamma)/\alpha} \int_\epsilon^t \epsilon^{\gamma/\alpha} F\left(\frac{V_u^{(\epsilon)}}{\epsilon^{1/\alpha}}\right) u^{-\beta} \, \mathrm{d}u.$$

For all $T > 0$,

$$\sup_{\epsilon \le t \le T} \left| V_t^{(\epsilon)} - L_t^{(\epsilon)} \right| \le \epsilon^{1/\alpha} |v_0 - L_1| + \epsilon^{\beta - 1 + (1-\gamma)/\alpha} \sup_{\epsilon \le t \le T} \left| \int_\epsilon^t \epsilon^{\gamma/\alpha} F\left(\frac{V_u^{(\epsilon)}}{\epsilon^{1/\alpha}}\right) u^{-\beta} \, \mathrm{d}u \right|.$$

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where $r = \min(\beta - 1 + 1/\alpha - p_\alpha(\gamma), 1/\alpha) > 0$.

# STEP 2B. CONVERGENCE OF $V$ FOR $\beta = 1 + \frac{\gamma-1}{\alpha}$

1. tightness (based on moment estimates).
2. convergence of f.d.d.
   - Change of time: $\left(\frac{V_{e^t}}{e^{t/\alpha}}\right)_{t\geq 1} = (H_t)_{t\geq 1}$, where $H$ is solution of
   $$\mathrm{d}H_s = \mathrm{d}R_s - \frac{H_s}{\alpha}\,\mathrm{d}s - e^{s\left(\frac{\gamma-1}{\alpha}+1-\beta\right)} F\left(H_s\right)\mathrm{d}s.$$
   - $\Lambda_F$-strong ergodicity (Kulik, 2009) + stationnarity of $H$:
   $$\left(\frac{V_{t_1}^{(\epsilon)}}{t_1^{1/\alpha}}, \frac{V_{t_2}^{(\epsilon)}}{t_2^{1/\alpha}}\right) \underset{\epsilon\to 0}{\Longrightarrow} \Lambda_{F,\log(t_1),\log(t_2)}.$$
   with $T : u := (u_1, u_2) \mapsto (t_1^{1/\alpha}u_1, t_2^{1/\alpha}u_2)$.

# Step 2b. Convergence of $V$ for $\beta = 1 + \frac{\gamma-1}{\alpha}$

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# Moment estimates $\alpha \in (1,2]$, $\gamma \in [0,1)$, $\kappa \in [0,1]$

Set $\tau_r = \inf\{t \geq t_0,\ |V_t| \geq r\}$, $r \geq 0$.
Jensen's inequality : $\mathbb{E}\left[|V_t|^\kappa\right] \leq \mathbb{E}\left[|V_t|\right]^\kappa$.

$$|V_{t\wedge\tau_r -}| \leq |v_0 - L_{t_0}| + |L_{t\wedge\tau_r -}| + \int_{t_0}^{t\wedge\tau_r} |F(V_{s\wedge\tau_r -})|\, s^{-\beta}\, \mathrm{d}s$$

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$$g_r(t) \leq g_r(t_0) + t^{\frac{1}{\alpha}}\,\mathbb{E}\left[|L_1|\right] + K\int_{t_0}^{t} g_r(s)^\gamma s^{-\beta}\,\mathrm{d}s$$

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$$|V_{t\wedge\tau_r-}| \leq |v_0 - L_{t_0}| + |L_{t\wedge\tau_r-}| + \int_{t_0}^{t\wedge\tau_r} K\,|V_{s\wedge\tau_r-}|^\gamma\, s^{-\beta}\,\mathrm{d}s$$

$$g_r(t) \leq g_r(t_0) + t^{\frac{1}{\alpha}}\mathbb{E}[|L_1|] + K\int_{t_0}^{t} g_r(s)^\gamma s^{-\beta}\,\mathrm{d}s$$

Gronwall-type lemma with $g_r : t \mapsto \mathbb{E}[|V_{t\wedge\tau_r-}|]$, $\frac{\gamma-1}{\alpha}+1 \leq \beta$ and Fatou's lemma:

$$\mathbb{E}[|V_{t\wedge\tau_\infty}|] \leq \liminf_{r\to\infty}\mathbb{E}[|V_{t\wedge\tau_r-}|] \leq C_{\gamma,\beta,t_0}\, t^{\frac{1}{\alpha}}.$$

# To go further

Complete the image of the convergence of the position process (work in progress):

- $q < \frac{1}{2}$ in the Brownian case,
- multidimensional case:

$$F : v \mapsto \mathrm{sgn}(v)\,|v|^\gamma \longleftrightarrow F : v \in \mathbb{R}^d = v\|v\|^{\gamma-1}$$

  - ✓ $q > 1/2$
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Thank you for your attention !

# References I

Patrick Cattiaux, Djalil Chafai, and Sébastien Motsch. “Asymptotic Analysis and Diffusion Limit of the Persistent Turning Walker Model”. In: *Asymptotic Analysis* 67.1–2 (2010), pp. 17–31.

Zhen-Qing Chen, Xicheng Zhang, and Guohuan Zhao. “Well-Posedness of Supercritical SDE Driven by Lévy Processes with Irregular Drifts”. In: *arXiv:1709.04632 [math]* (Sept. 2017).

Chang-Song Deng and René L. Schilling. “On Shift Harnack Inequalities for Subordinate Semigroups and Moment Estimates for Lévy Processes”. In: *Stochastic Processes and their Applications* 125.10 (Oct. 2015), pp. 3851–3878.

# References II

Richard Eon and Mihai Gradinaru. "Gaussian Asymptotics for a Non-Linear Langevin Type Equation Driven by an *alpha*-Stable Lévy Noise". In: *Electronic Journal of Probability* 20 (2015).

Nicolas Fournier and Camille Tardif. "One Dimensional Critical Kinetic Fokker-Planck Equations, Bessel and Stable Processes". In: *arXiv:1805.09728 [math]* (May 2018).

Mihai Gradinaru and Yoann Offret. "Existence and Asymptotic Behaviour of Some Time-Inhomogeneous Diffusions". In: *Annales de l'IHP, PS* 49.1 (Feb. 2013), pp. 182–207.

# References III

Alexey M. Kulik. "Exponential Ergodicity of the Solutions to SDE's with a Jump Noise". In: *Stochastic Processes and their Applications* 119.2 (Feb. 2009), pp. 602–632.