Institut de Mathématiques de Bordeaux

université de BORDEAUX

La marche aléatoire de l'éléphant et son centre de masse

Journées de Probabilités 2021

Lucile Laulin

21 juin 2021

1. The Elephant Random Walk
2. The Multidimensional Elephant Random Walk
3. The Center of Mass

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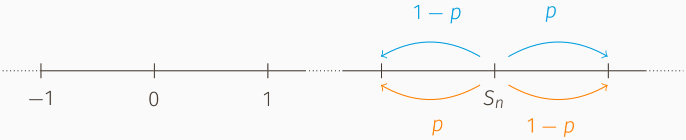

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A martingale approach

We can write $X_{n+1} = \alpha_{n+1} X_{\beta_{n+1}}$ where the random variables

$$\alpha_{n+1} \sim \mathcal{R}(p), \qquad \beta_{n+1} \sim \mathcal{U}(1, \ldots, n)$$

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Then,

$$\begin{aligned}
\mathbb{P}(X_{n+1} = 1 | \mathcal{F}_n) &= p \frac{\#\{\text{steps to the right}\}}{n} + (1-p) \frac{\#\{\text{steps to the left}\}}{n} \\
&= p \frac{S_n + n}{2n} + (1-p) \frac{n - S_n}{2n} \\
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The conditionnal distribution of X_{n+1} given the past is

$$\mathcal{L}(X_{n+1} | \mathcal{F}_n) = \mathcal{R}(p_n)$$

where $p_n = \frac{1}{2}\left(1 + a\frac{S_n}{n}\right)$ and $a = 2p - 1$.

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We deduce that

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1}|\mathcal{F}_n] = S_n + (2p_n - 1) = \Big(1 + \frac{a}{n}\Big)S_n = \gamma_n S_n.$$

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The study relies on a martingale approach

$$M_n = a_n S_n$$

where $a_1 = 1$ and

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The process (M_n) is a locally bounded square-integrable martingale. Indeed,

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = a_{n+1}\mathbb{E}[S_{n+1}|\mathcal{F}_n] = a_{n+1}\gamma_n S_n = a_n S_n = M_n$$

and $\mathbb{E}[M_n^2] \leq (na_n)^2$.

A martingale approach

The process (M_n) can be rewritten as

$$M_n = \sum_{k=1}^{n} a_k \varepsilon_k \quad \text{where} \quad \varepsilon_k = S_k - \gamma_{k-1} S_{k-1}.$$

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Let $\Delta M_k = M_k - M_{k-1} = a_k \varepsilon_k$. The asymptotical behavior of (M_n) is directly related to the one of its quadratic variation defined as

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We need to compute

$$\begin{aligned}
\mathbb{E}[\varepsilon_k^2 | \mathcal{F}_{k-1}] &= \mathbb{E}[S_k^2 | \mathcal{F}_{k-1}] - (\gamma_{k-1} S_{k-1})^2 \\
&= \mathbb{E}[S_{k-1}^2 + 2 S_{k-1} X_k + 1 | \mathcal{F}_{k-1}] - (\gamma_{k-1} S_{k-1})^2 \\
&= S_{k-1}^2 + 2(\gamma_{k-1} - 1) S_{k-1}^2 + 1 - (\gamma_{k-1} S_{k-1})^2 \\
&= 1 - (\gamma_{k-1} - 1)^2 S_k^2.
\end{aligned}$$

Three regimes

Hence, we obtain that

$$\langle M \rangle_n = \sum_{k=1}^{n} a_k^2 - a^2 \sum_{k=1}^{n} a_k^2 \Big(\frac{S_k}{k}\Big)^2.$$

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- the critical regime where $a = 1/2$ and $v_n = O(\log n)$,
- the superdiffusive regime where $a > 1/2$ and $v_n = O(1)$.

Main results

Theorem (Law of large numbers)

Diffusive	*Critical*	*Superdiffusive*
$\lim_{n\to\infty} \frac{S_n}{n} \overset{a.s.}{=} 0$	$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} \overset{a.s.}{=} 0$	$\lim_{n\to\infty} \frac{S_n}{n^a} \overset{a.s./\mathbb{L}^4}{=} L$

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Theorem (Quadratic strong law and law of iterated logarithm)

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$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\frac{S_k}{k}\right)^2 \overset{a.s.}{=} \frac{1}{1-2a}$$

$$\limsup_{n\to\infty} \frac{S_n^2}{2n\log\log n} \overset{a.s.}{=} \frac{1}{1-2a}$$

Critical

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \left(\frac{S_k}{k\log k}\right)^2 \overset{a.s.}{=} 1$$

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Theorem (Asymptotic normality)

Diffusive

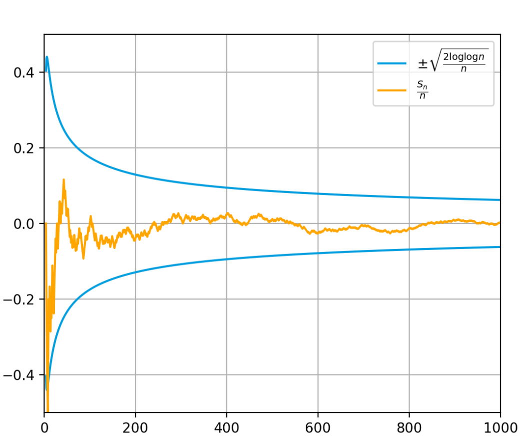
$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{1-2a}\right)$$

Critical

$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$$

Superdiffusive

$$\frac{S_n - n^a L}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{2a-1}\right)$$


$\pm\sqrt{\frac{2\log\log n}{n}}$
$\frac{S_n}{n}$
0.4
0.2
0.0
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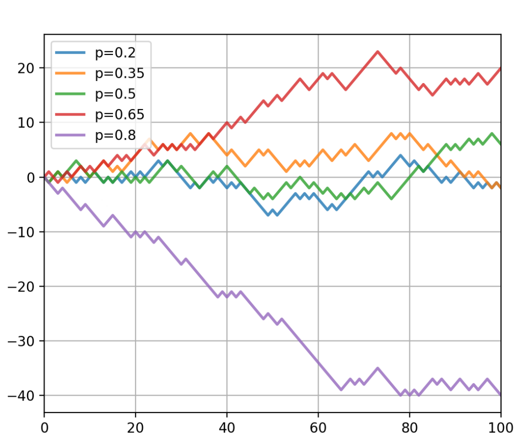
200
400
600
800
1000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p=0.35
p=0.5
p=0.65
p=0.8
20
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The Multidimensional Elephant Random Walk

The multidimensional ERW

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with $A_{n+1} \in \mathcal{M}_d(\mathbb{R})$ such that

$$A_{n+1} = \begin{cases} +I_d \text{ with probability } p \\ -I_d \text{ with probability } \frac{1-p}{2d-1} \\ +J_d \text{ with probability } \frac{1-p}{2d-1} \\ -J_d \text{ with probability } \frac{1-p}{2d-1} \\ \quad \vdots \\ +J_d^{d-1} \text{ with probability } \frac{1-p}{2d-1} \\ -J_d^{d-1} \text{ with probability } \frac{1-p}{2d-1} \end{cases} \quad \text{and} \quad J_d = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad J_d^d = I_d.$$

The multidimensional ERW

The critical parameter is

$$p_d = \frac{2d+1}{4d} \quad \Longleftrightarrow \quad a = \frac{2dp-1}{2d-1}.$$

Once again, there are three regimes of behavior.

Remark. The case of the usual random walk where $p = \dfrac{1}{2d}$ always appears in the diffusive regime.

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Theorem (Quadratic strong law and law of iterated logarithm)

Diffusive

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\frac{S_k S_k^T}{k}\right)^2 \overset{a.s.}{=} \frac{1}{(1-2a)d} I_d$$

$$\limsup_{n\to\infty} \frac{\|S_n\|^2}{2n\log\log n} \overset{a.s.}{=} \frac{1}{(1-2a)d}$$

Critical

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \left(\frac{S_k S_k^T}{k\log k}\right)^2 \overset{a.s.}{=} \frac{1}{d} I_d$$

$$\limsup_{n\to\infty} \frac{\|S_n\|^2}{2n\log n\log\log\log n} \overset{a.s.}{=} \frac{1}{d}$$

Main results for the MERW

Theorem (Law of large numbers)

Diffusive

$$\lim_{n\to\infty} \frac{\|S_n\|}{n} \overset{a.s.}{=} 0$$

Critical

$$\lim_{n\to\infty} \frac{\|S_n\|}{\sqrt{n}\log n} \overset{a.s.}{=} 0$$

Superdiffusive

$$\lim_{n\to\infty} \frac{\|S_n\|}{n^a} \overset{a.s./\mathbb{L}^2}{=} L_d$$

Theorem (Quadratic strong law and law of iterated logarithm)

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$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\frac{S_k S_k^T}{k}\right)^2 \overset{a.s.}{=} \frac{1}{(1-2a)d} I_d$$

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Theorem (Asymptotic normality)

Diffusive

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{(1-2a)d} I_d\right)$$

Critical

$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{d} I_d\right)$$

The Center of Mass

Center of mass of a random walk

The center of mass of a random walk $(S_n)_n$ is defined by

$$G_n = \frac{1}{n}\sum_{k=1}^{n} S_k.$$

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In the usual case, where (X_n) are i.i.d random vectors of $\mathbb{R}^d$ with mean $\boldsymbol{\mu}$ and covariance matrix Γ, it is true that

$$G_n = \frac{1}{n} \sum_{k=1}^{n} S_k = \frac{1}{n} \sum_{k=1}^{n} (n-k+1) X_k \quad \text{and} \quad \Sigma_n = \frac{1}{n} \sum_{k=1}^{n} k X_k$$

share the same distribution.

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share the same distribution.

It is possible to obtain the strong law of large numbers and the asymptotic normality.

Theorem (Lo and Wade, 2019)

$$\lim_{n\to\infty}\frac{1}{n}G_n = \frac{1}{2}\boldsymbol{\mu} \ a.s. \quad and \quad \frac{1}{\sqrt{n}}\Big(G_n - \frac{n}{2}\boldsymbol{\mu}\Big) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3d}\Gamma\Big)$$

Center of mass of the ERW

In the case of the MERW on $\mathbb{Z}^d$, we have

$$
\begin{aligned}
G_n &= \frac{1}{n}\sum_{k=1}^{n} S_k = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}M_k \\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\sum_{\ell=1}^{k} a_\ell \varepsilon_\ell = \frac{1}{n}\sum_{k=1}^{n} a_k\varepsilon_k \sum_{\ell=k}^{n}\frac{1}{a_\ell}, \\
&= \frac{1}{n}\sum_{k=1}^{n} a_k(b_n - b_{k-1})\varepsilon_k
\end{aligned}
$$

with $b_0 = 0$ and $b_n = \sum_{k=1}^{n} a_k^{-1}$.

Center of mass of the ERW

In the case of the MERW on $\mathbb{Z}^d$, we have

$$\begin{aligned} G_n &= \frac{1}{n}\sum_{k=1}^{n} S_k = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}M_k \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\sum_{\ell=1}^{k} a_\ell \varepsilon_\ell = \frac{1}{n}\sum_{k=1}^{n} a_k\varepsilon_k \sum_{\ell=k}^{n}\frac{1}{a_\ell}, \\ &= \frac{1}{n}\sum_{k=1}^{n} a_k(b_n - b_{k-1})\varepsilon_k \end{aligned}$$

with $b_0 = 0$ and $\;b_n = \sum_{k=1}^{n} a_k^{-1}.$

Consequently, G_n can be rewritten

$$G_n = \frac{1}{n}\big(b_n M_n - N_n\big) \quad \text{where} \quad N_n = \sum_{k=1}^{n} a_k b_{k-1}\varepsilon_k.$$

A martingale approach

The process (N_n) is a square integrable martingale such that

$$\langle N \rangle_n = w_n + o(w_n) = O(n^3) \quad \text{a.s.}$$

where $w_n = \sum_{k=1}^{n} a_k^2 b_{k-1}^2$.

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⚠ The quadratic variations of M_n and N_n increase at different speeds.

→ The solution is to study the multivariate martingale $\mathcal{M}_n = (M_n \;\; N_n)^T \in \mathbb{R}^{2d}$ and to use a matrix normalization

$$V_n = \frac{1}{n\sqrt{n}} \begin{pmatrix} b_n & 0 \\ 0 & 1 \end{pmatrix} \otimes I_d = \frac{1}{n\sqrt{n}} \begin{pmatrix} b_n I_d & 0 \\ 0 & I_d \end{pmatrix}$$

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The process (N_n) is a square integrable martingale such that

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in the way that

$$\frac{G_n}{\sqrt{n}} = v^T V_n \mathcal{M}_n \quad \text{where} \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes I_d.$$

Usefull results

Theorem (Touati, 1991)

Let $(\mathcal{M}_n)$ be a locally square-integrable martingale of $\mathbb{R}^\delta$ adapted to a filtration $(\mathcal{F}_n)$, with predictable quadratic variation $\langle \mathcal{M} \rangle_n$. Let (V_n) be a sequence of non-random square matrices of order δ such that $\|V_n\|$ decreases to 0 as n goes to infinity. Assume that there exists a symmetric and positive semi-definite matrix V such that

$$\text{(H.1)} \qquad V_n \langle \mathcal{M} \rangle_n V_n^T \xrightarrow[n\to\infty]{\mathbb{P}} V.$$

Moreover, assume that Lindeberg's condition is satisfied, that is for all $\varepsilon > 0$,

$$\text{(H.2)} \qquad \sum_{k=1}^{n} \mathbb{E}\left[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \middle| \mathcal{F}_{k-1}\right] \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

where $\Delta \mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$. Then, we have the asymptotic normality

$$V_n \mathcal{M}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, V).$$

Usefull results

Theorem (Chaâbane and Maâouia, 2000)

Let $(\mathcal{M}_n)$ be a locally square-integrable martingale of $\mathbb{R}^\delta$ adapted to a filtration $(\mathcal{F}_n)$, with predictable quadratic variation $\langle \mathcal{M} \rangle_n$. Let (V_n) be a sequence of non-random positive definite diagonal matrices of order δ such that its diagonal terms decrease to zero at polynomial rates. Assume that (H.1) *and* (H.2) *hold almost surely. Moreover, suppose that there exists $\beta \in]1, 2]$ such that*

$$\text{(H.3)} \qquad \sum_{n=1}^{\infty} \frac{1}{\left(\log(\det V_n^{-1})^2\right)^{\beta}} \mathbb{E}\left[\|V_n \Delta \mathcal{M}_n\|^{2\beta} \middle| \mathcal{F}_{n-1} \right] < \infty \qquad a.s.$$

Then, we have the quadratic strong law

$$\lim_{n \to \infty} \frac{1}{\log(\det V_n^{-1})^2} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2} \Big) V_k \mathcal{M}_k \mathcal{M}_k^T V_k^T = V \qquad a.s.$$

Main results

Theorem (Law of large numbers)

Diffusive

$$\lim_{n\to\infty} \frac{S_n}{n} \overset{a.s.}{=} 0$$

Critical

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} \overset{a.s.}{=} 0$$

Superdiffusive

$$\lim_{n\to\infty} \frac{S_n}{n^a} \overset{a.s./\mathbb{L}^4}{=} L$$

Theorem (Quadratic strong law and law of iterated logarithm)

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$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{S_k S_k t^T}{k}\Big)^2 \overset{a.s.}{=} \frac{1}{(1-2a)d} I_d$$

$$\limsup_{n\to\infty} \frac{\|S_n\|^2}{2n\log\log n} \overset{a.s.}{=} \frac{1}{(1-2a)d}$$

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$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \Big(\frac{S_k S_k^T}{k\log k}\Big)^2 \overset{a.s.}{=} \frac{1}{d} I_d$$

$$\limsup_{n\to\infty} \frac{\|S_n\|^2}{2n\log n\log\log\log n} \overset{a.s.}{=} \frac{1}{d}$$

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$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{(1-2a)d} I_d\Big)$$

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$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, \frac{1}{d} I_d\big)$$

Main results

Theorem (Law of large numbers)

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$$\lim_{n\to\infty} \frac{G_n}{n} \overset{a.s.}{=} 0$$

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$$\lim_{n\to\infty} \frac{G_n}{\sqrt{n}\log n} \overset{a.s.}{=} 0$$

Superdiffusive

$$\lim_{n\to\infty} \frac{G_n}{n^a} \overset{a.s./\mathbb{L}^2}{=} \frac{L}{a+1}$$

Theorem (Quadratic strong law and law of iterated logarithm)

Diffusive

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\frac{G_k G_k^T}{k}\right)^2 \overset{a.s.}{=} \frac{2}{3(1-2a)(2-a)d} I_d$$

Critical

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \left(\frac{G_k G_k^T}{k \log k}\right)^2 \overset{a.s.}{=} \frac{4}{9d} I_d$$

$$\limsup_{n\to\infty} \frac{\|G_n\|^2}{2n \log n \log\log\log n} \overset{a.s.}{=} \frac{4}{9d}$$

Theorem (Asymptotic normality)

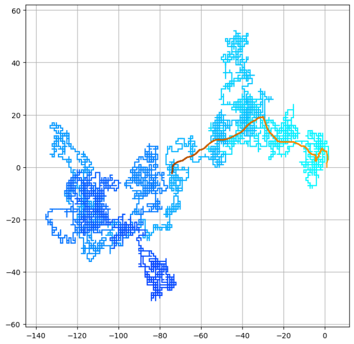
Diffusive

$$\frac{G_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{2}{3(1-2a)(2-a)d} I_d\right)$$

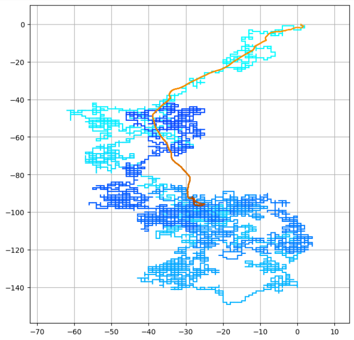
Critical

$$\frac{G_n}{\sqrt{n \log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{4}{9d} I_d\right)$$

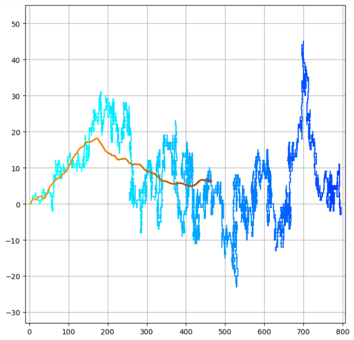
Trajectories when d=2



$p = 0.30$



$p = 0.55$



$p = 0.70$

Merci pour votre attention !