Repulsion of zeros of Gaussian fields

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Point Process

• **A point process** is defined as a mapping from a probability space to configurations of points of $\mathbb{R}^d$, namely its intersection with any compact of $\mathbb{R}^d$ is locally finite.

▷ Points processes have emerged as powerful tools for modeling natural phenomena such as monitoring a population or plants locations.

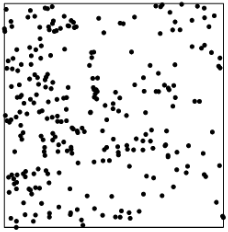

FIGURE 1 – Position of 271 adult pine trees in a forest (MA.Gallego, V.Ibañez, A.Simo "Parameter estimation in non-homogeneous Boolean models : an application to plant defense response." Image Analysis and Stereology)

Repulsion and Attraction

One of the central questions in the study of point process is to know if the points have a tendency to attract or repel each other.
Repulsive point processes are useful in a number of applications, such as sampling points for quasi Monte Carlo methods, data mining, training set selection in machine learning.

Example :

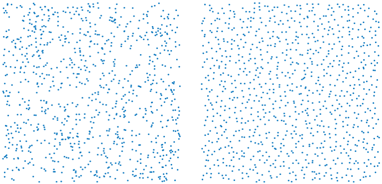

FIGURE 2 – Left : The Poisson point process. Right : The Determinantal point process (D.Beliaev, V.Cammarota, I.Wigman)

Repulsion and Attraction

Definition

For X a point process and $\mathcal{B}_\rho$ a small disc of radius $\rho > 0$.
As $\rho \to 0$

- Local Repulsion :

$$\mathbb{P}(\#X \cap \mathcal{B}_\rho = 2) << (\mathbb{P}(\#X \cap \mathcal{B}_\rho = 1))^2$$

- Local Attraction :

$$(\mathbb{P}(\#X \cap \mathcal{B}_\rho = 1))^2 << \mathbb{P}(\#X \cap \mathcal{B}_\rho = 2)$$

Remark

For Poisson process (or a finite collection of independent points) on the plane : $(\#X \cap \mathcal{B}_\rho) \overset{\mathcal{L}}{=} \mathit{Poiss}(\lambda\pi\rho^2)$

$$\mathbb{P}(\#X \cap \mathcal{B}_\rho = 2) \approx \rho^4 \qquad \mathbb{P}(\#X \cap \mathcal{B}_\rho = 1) \approx \rho^2.$$

Gaussian random Field

A centered random field $\psi : \mathbb{R}^2 \to \mathbb{R}$ is a centered Gaussian random field, if any finite linear combination of all these random vectors are Gaussian. The distribution of a Gaussian random field is determined by its expectation and covariance functions :

- The covariance function of ψ is defined by

$$\begin{array}{rccc} \sigma : & \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (s,t) & \mapsto & \mathbb{E}\left[\psi(s)\psi(t)\right] \end{array}$$

Gaussian random Field

- ψ is said stationary, if its law is invariant under the translation :

$$\forall (s,t) \in (\mathbb{R}^2)^2, \quad \sigma(s,t) = \sigma(s-t,0)$$

- ψ is said isotropic, if its law is invariant under any rotation θ of the parameter space :

$$\forall \theta \in \mathrm{SO}(2),\ \forall (s,t) \in (\mathbb{R}^2)^2, \quad \sigma(\theta(s),\theta(t)) = \sigma(s,t).$$

- If ψ is sationary and isotropic, one can write σ by

$$\sigma(s,t) = \sigma(\|z-w\|)$$

Example of covariance function :

- $\sigma(x) = \exp(-\frac{x^2}{2}), \quad x \in \mathbb{R}$
- $\sigma(x) = J_0(x) \quad$ where $= J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin(x))dx$ is the Bessel function.

Critical points

Study of critical points : number ?, position ?, repulsion behavior ?
are important issues in various application areas such as detection of peaks of the random field and the geometry of nodal lines . . . .
For a smooth Gaussian field $\psi : \mathbb{R}^2 \to \mathbb{R}$, stationary, isotropic, the number of critical points in a small ball $\mathcal{B}_\rho$ of radius $\rho > 0$

$$\mathcal{N}_\rho^c(\psi) = \#\{x \in \mathcal{B}_\rho : \nabla\psi(x) = 0\}.$$

- ψ smooth Gaussian $\Rightarrow$ set of critical points is a point process on $\mathbb{R}^2$.

D.Belyaev, V.Cammarota and I.Wigman(2019)

They study the repulsion of the critical points for a particular Gaussian field $\psi : \mathbb{R}^2 \to \mathbb{R}$ the "Berry's Planar Random Wave Model"(covariance function Bessel).
Neutral repulsion between the critical points and a strong repulsion between minima and maxima is showed.

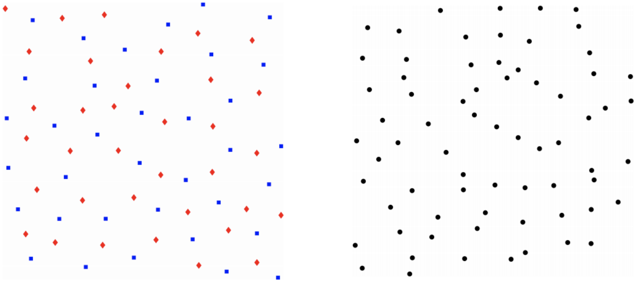

FIGURE 3 – Left : extrema only (red diamonds are local maxima, blue squares are local minima), right : saddles only.

Model

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^4$ smooth Gaussian field, stationary, isotropic and centered with variance 1. The law of ψ is prescribed by the covariance function

$$\mathbb{E}[\psi(z)\psi(w)] := \sigma(\|z - w\|) \qquad z, w \in \mathbb{R}^2.$$

- We study the local repulsion of critical points and study separately the different types of critical points (local max, local min, saddles).
- To quantify the repulsion of the critical points process, we compute

$\bullet\ \mathbb{E}[\mathcal{N}_\rho^c(\psi)] = \mathbb{P}(\mathcal{N}_\rho^c(\psi) = 1) + 2\mathbb{P}(\mathcal{N}_\rho^c(\psi) = 2) + 3\mathbb{P}(\mathcal{N}_\rho^c(\psi) = 3) + \dots$

$\bullet\ \mathbb{E}[\mathcal{N}_\rho^c(\psi)(\mathcal{N}_\rho^c(\psi) - 1)] = 2\mathbb{P}(\mathcal{N}_\rho^c(\psi) = 2) + 6\mathbb{P}(\mathcal{N}_\rho^c(\psi) = 3) + \dots$

Kac-Rice formula

We assume that the Gaussian vector $(\nabla\psi(z_1)\ldots,\nabla\psi(z_k))$ is non-degenerate, $\forall z_i \in \mathbb{R}^2$. Then, for $n \geqslant 1$, the n-th factorial moment of $\mathcal{N}^c_\rho(\psi)$ is given by

$$\mathbb{E}[\mathcal{N}^c_\rho(\mathcal{N}^c_\rho - 1)\ldots(\mathcal{N}^c_\rho - (n-1))] = \int\ldots\int_{\mathcal{B}(\rho)\times\ldots\times\mathcal{B}(\rho)} \mathrm{K}_n(z)\,\mathrm{d}z,$$

where $z = (z_1\ldots,z_n) \in \mathcal{B}(\rho)\times\ldots\times\mathcal{B}(\rho) \subset \mathbb{R}^{2n}$,
and K_n is the n-point correlation function

$$\mathrm{K}_n(z) = \phi_{(\nabla\psi(z_1),\ldots,\nabla\psi(z_n))}(0,\ldots,0)\times \mathbb{E}\left[\prod_{i=1}^n |\det H_\psi(z_i)| \,\Big|\, \nabla\psi(z_1) = \ldots = \nabla\psi(z_n) = 0\right],$$

where $\phi_{(\nabla\psi(z_1),\ldots,\nabla\psi(z_n))}(0,\ldots,0)$ is the density probability function of the Gaussian vector $(\nabla\psi(z_1),\ldots,\nabla\psi(z_n))$ evaluated at $(0,\ldots,0)$, and $H_\psi(z_i)$ is the Hessian matrix at z_i.

Expected number of critical points

First Result :

For every $\rho > 0$, we have

$$\mathbb{E}[\mathcal{N}_\rho^c] = c\ \rho^2 \quad (\mathbb{E}(\#\text{critical points})),$$
$$\mathbb{E}[\mathcal{N}_\rho^e] = c_e\ \rho^2 \quad (\mathbb{E}(\#\text{extremal points})),$$
$$\mathbb{E}[\mathcal{N}_\rho^s] = c_s\ \rho^2 \quad (\mathbb{E}(\#\text{saddles points})),$$
$$\mathbb{E}[\mathcal{N}_\rho^{min}] = \mathbb{E}[\mathcal{N}_\rho^{max}] = \frac{1}{2}\mathbb{E}[\mathcal{N}_\rho^e].$$

We have exact values for c, c_e and c_s, only depending on $\sigma'(0), \sigma''(0)$.

The second factorial moment of $\mathcal{N}_\rho^c, \mathcal{N}_\rho^e, \mathcal{N}_\rho^s$

Theorem

As $\rho \to 0$, we have the following asymptotic

$$\mathbb{E}[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)] \sim a_c \, \rho^4.$$

For the numbers of extrema, saddles in a ball of radius ρ, we have

$$\mathbb{E}[\mathcal{N}_\rho^e(\mathcal{N}_\rho^e - 1)] \asymp \rho^7,$$
$$\mathbb{E}[\mathcal{N}_\rho^s(\mathcal{N}_\rho^s - 1)] \asymp \rho^7,$$
$$\mathbb{E}[\mathcal{N}_\rho^e \mathcal{N}_\rho^s] \sim a_{e,s} \, \rho^4$$

where $a_c, a_{e,s}$ are positive constants.

Repulsion or not ?

As $\rho \to 0$,

$$\begin{array}{ll} \mathbb{P}(\mathcal{N}_\rho^c = 1) \asymp \rho^2 & \mathbb{P}(\mathcal{N}_\rho^c = 2) \asymp \rho^4 \\ \mathbb{P}(\mathcal{N}_\rho^e = 1) \asymp \rho^2 & \mathbb{P}(\mathcal{N}_\rho^e = 2) \asymp \rho^7 \\ \mathbb{P}(\mathcal{N}_\rho^s = 1) \asymp \rho^2 & \mathbb{P}(\mathcal{N}_\rho^s = 2) \asymp \rho^7 \end{array}$$

- Critical points : exhibit no repulsion no attraction.
- Extrema, Saddles : local repulsion.

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Repulsion or not ?

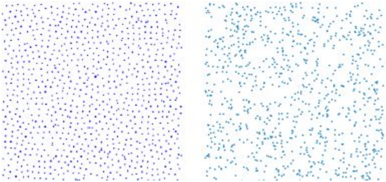

FIGURE 4 – Left : critical points of a random plane wave. Right : The Poisson point process which has the same density. (D. Beliaev, V. Cammarota, I. Wigman)

Similar work

- Azais and Delmas 2019 : They study the attraction or repulsion of critical points for general stationary Gaussian fields in any dimension. Using a different method(random Matrix Theory)
  Critical points :
  - $N > 2$: attraction,
  - $N = 2$: neutrality.

  $\forall N$: a strong repulsion between maxima and minima
  $(\mathbb{P}(\mathcal{N}_\rho^{max} \geqslant 1, \mathcal{N}_\rho^{min} \geqslant 1,) << \mathbb{P}(\mathcal{N}_\rho^{max} \geqslant 1)\mathbb{P}(\mathcal{N}_\rho^{min} \geqslant 1))$

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Thank you