Bornes de la solution de l'équation de Stein relative aux lois gaussiennes inverses généralisées et de Kummer

Angelo Efoévi Koudou (Univ. de Lorraine)

Journées de probabilités
Guidel, 25 juin 2021

Outline

1. The generalized inverse Gaussian distribution

The generalized inverse Gaussian distribution with parameters $p \in \mathbb{R}$, $a > 0$, $b > 0$, which will be denoted $\mathrm{GIG}(p, a, b)$, has density

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0$$

where K_p is the modified Bessel function of the third kind:

$$K_p(z) = 2^{-p-1} z^p \int_0^\infty x^{-p-1} e^{-x-\frac{z^2}{4x}} dx, \quad \mathrm{Re}(z) > 0.$$

If $p < 0$, then $\mathrm{GIG}(p, a, b)$ converges to a gamma distribution as $b \to 0$.
If $p > 0$, then $\mathrm{GIG}(p, a, b)$ converges to a reciprocal gamma distribution as $a \to 0$.

- if $W \sim GIG(p, a, b)$ then

$$\mathbb{E}(W) = \sqrt{\frac{b}{a}} \frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}, \qquad \mathbb{E}(W^2) = \frac{b}{a} \frac{K_{p+2}(\sqrt{ab})}{K_p(\sqrt{ab})}.$$

- If $X \sim \mathrm{GIG}(p, a, b)$ then $\frac{1}{X} \sim \mathrm{GIG}(-p, b, a)$.
- For p, a, $b > 0$, if $X \sim \mathrm{GIG}(-p, a, b)$, $Y \sim \gamma(p, \frac{a}{2})$, and X and Y are independent, then $X + Y \sim \mathrm{GIG}(p, a, b)$, where $\gamma(p, c) = \mathrm{GIG}(p, 2c, 0)$ is the gamma distribution with density $\frac{c^p}{\Gamma(p)} x^{p-1} \exp -cx \quad (x > 0)$. (Barndorff-Nielsen and Halgreen, 1977).

- The name "generalized inverse Gaussian" was proposed by Good (1953) in his study of population frequencies. But this three-parameter law appears in a work by Halphen (1941), not signed under his own name, probably because of the war context. For this reason the GIG law is also called "Halphen type A distribution".

Some examples of use of the GIG distribution

- Jorgensen (1982) proved a better fit of GIG than the exponential distribution to data consisting in
  - intervals (in hours) between successive failures of airconditioning equipment in Boeing 720 aircraft;
  - intervals between pulses along a nerve fibre;
  - intervals between the times at which vehicles pass a point on a road.
- Iyengar & Liao (1997) : neural activity (interspike intervals for neurons); comparison of the GIG fit with the lognormal fit.
- Chebana *et al* (2010) : application to extreme hydrologic events

A few words about the case $p = -\frac{1}{2}$

If $p = -\frac{1}{2}$, then $GIG(p, a, b)$ is the *inverse Gaussian (IG) distribution* :

$$IG(a, b)(dx) = \sqrt{\frac{b}{2\pi}} e^{\sqrt{ab}} x^{-\frac{3}{2}} e^{-\frac{1}{2}(ax+b/x)} \mathbf{1}_{(0,\infty)}(x) dx.$$

- The history of IG dates back to 1915 when Schrödinger and Smoluchowski obtained, independently, the density of the first passage time of Brownian motion with positive drift. The drift-free case had already been treated by Bachelier (1900) in his thesis on the theory of speculation.
- This distribution was named "inverse Gaussian" by Tweedie (1945), who observed that the cumulant function of this law is the inverse of the cumulant function of the normal law.

- The inverse Gaussian distribution is used in data analysis when the observations are highly right-skewed, e.g. in cardiology, demography, finance, biology, hydrology, pharmacokinetics. See Chhikara and Folks (1989), Seshadri (1999).
- If $p = \frac{1}{2}$ we have the reciprocal inverse Gaussian distribution

$$\mathrm{RIG}(a, b)(dx) = \sqrt{\frac{a}{2\pi}} e^{\sqrt{ab}} x^{-\frac{1}{2}} e^{-\frac{1}{2}(ax + b/x)} \mathbf{1}_{(0,\infty)}(x) dx.$$

- IG and RIG laws are respectively the distribution of the first and the last hitting time for a Brownian motion (cf e.g. Bhattacharya and Waymire, 1990).
  For a standard Brownian motion B, define, for $a \geq 0$ and $b > 0$,
  $$\tau_b^a = \inf\{t > 0;\ B_t + \sqrt{b}t = a\},$$
  $$\sigma_b^a = \sup\{t > 0;\ B_t + \sqrt{b}t = a\}.$$
  Then $\tau_b^a \sim \mathrm{IG}(a, b)$ and $\sigma_b^a \sim \mathrm{RIG}(a, b)$.
- More generally, depending on the sign of p, GIG(p, a, b) can be viewed as the distributions of either first or last exit times of certain diffusion processes (see Barndorff-Nielsen *et al* (1978) and also Vallois (1991) considering Bessel processes with drift).

- The GIG density can be defined on the set of positive definite matrices, the case $a = 0$ defining Wishart matrices (see Letac and Wesolowski, 2000).
- The Black-Scholes formula in finance can be expressed in terms of the distribution function of GIG variables (see Madan, Roynette and Yor, 2008).

The Matsumoto-Yor property

Let $p > 0$, $a > 0$ *and* $b > 0$. *Consider two independent, positive random variables* X *and* Y *such that*

$$X \sim \mathrm{GIG}(-p, a, b), \quad Y \sim \gamma(p, b^2/2) = \mathrm{GIG}(p, 0, b).$$

Then the random variables

$$U = \frac{1}{X+Y}, \quad V = \frac{1}{X} - \frac{1}{X+Y}$$

are independent.

Proof by Matsumoto and Yor (*Nagoya Math. J.*, 2001) in the case $a = b$.

Letac and Wesołowski (*Ann. of Prob.* , 2000) noticed that it is true also if $a \neq b$ and proved that property to be in fact a characterization of GIG laws:

Theorem

(Letac and Wesołowski, 2000)
Consider two non-Dirac, positive and independent random variables X *and* Y. *Then the random variables* $U = (X + Y)^{-1}$ *and* $V = X^{-1} - (X + Y)^{-1}$ *are independent if and only if there exist* $p > 0$, $a > 0$ *and* $b > 0$ *such that* $X \sim \mathrm{GIG}(-p, a, b)$ *while* $Y \sim \gamma(p, b^2/2)$.

A generalization

For $f : (0, \infty) \to (0, \infty)$ a decreasing and bijective function, define the transformation

$$T_f : (0, \infty)^2 \to (0, \infty)^2$$

$$(x, y) \mapsto (f(x + y),\ f(x) - f(x + y)).$$

Questions:

- Are there other functions f than $x \mapsto 1/x$ such that T_f preserves the independence of some random variables X and Y ?
- Can we find all such f's?
- For each f, what are the laws of X and Y ?

We call such functions *Matsumoto-Yor functions*.
We have solved this questions under the assumption that f is smooth and that X and Y have smooth density functions.

Characterization of smooth Matsumoto-Yor functions

Let us introduce, for $x > 0$,

$$f_1(x) = \frac{1}{e^x - 1}, \quad g_1(x) = f_1^{-1}(x) = \log\left(1 + \frac{1}{x}\right),$$

and, for $\delta > 0$,

$$j_\delta(x) = \log\left(\frac{e^x + \delta - 1}{e^x - 1}\right).$$

Theorem

(K., Vallois 2012) Under some smothness assumptions, f is a Matsumoto-Yor function if and only if

$$f(x) = \frac{\alpha}{x}, \quad f(x) = \frac{1}{\alpha} f_1(\beta x),$$

$$f(x) = \frac{1}{\alpha} g_1(\beta x) \ \text{ or } \ f(x) = \frac{1}{\alpha} j_\delta(\beta x)$$

for some $\alpha, \beta, \delta > 0$.

A functional differential equation

Proposition

Let $X, Y > 0$ be two independent rv's. Suppose the densities p_X and p_Y of X and Y are positive and twice differentiable. Let $\phi_X = \log p_X$ and $\phi_Y = \log p_Y$. Consider a decreasing function $f : (0,\infty) \mapsto (0,\infty)$, three times differentiable.
The rv's $U = f(X+Y)$ and $V = f(X) - f(X+Y)$ are independent iff, for all $x, y > 0$:

$$\phi''_X(x) - \phi'_X(x)\frac{f''(x)}{f'(x)} + \phi''_Y(y) f'(x)\left(\frac{1}{f'(x)} - \frac{1}{f'(x+y)}\right)$$
$$+\phi'_Y(y)\frac{f''(x)}{f'(x)} + \frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = 0.$$

If we replace f with g_1 and solve the previous equation, then the Kummer distribution appears as the law of X.

2. The Kummer distribution

(see for instance Ng and Kotz, 1995).
For $a > 0$, $b \in \mathbb{R}$, $c > 0$, the Kummer distribution $\mathrm{K}(a, b, c)$ with parameters a, b, c has density

$$k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^{a-1}(1+x)^{-a-b}e^{-cx}, \quad x > 0$$

where ψ is the confluent hypergeometric function of the second kind. If $Z \sim K(a, b, c)$, then

$$\mathbb{E}(Z) = a\frac{\psi(a+1, 2-b; c)}{\psi(a, 1-b; c)}; \qquad \mathbb{E}(Z^2) = a(a+1)\frac{\psi(a+2, 3-b; c)}{\psi(a, 1-b; c)}.$$

- P. Vallois (unpublished work, 2012): if $G \sim \gamma(a, c)$ and W uniform on $(0, 1)$, independent of G, then $K(a, b, c)$ is the law of G conditional on $W(1+G)^{a+b} < 1$.
- $K(a, b, c)$ appears as the law of some random continued fractions in Marklov *et al* (2008) studying invariant measures for products of random matrices.

Characterization of the product of gamma and Kummer distributions by a Matsumoto-Yor type independence property

Theorem

(Piliszek and Wesołowski, 2018)

Let $U, V > 0$ be two independent rv's. The r.v.'s U' and V' defined by

$$(U', V') = \left(\frac{1 + \frac{1}{U+V}}{1 + \frac{1}{U}}, \; U + V \right)$$

are independent if only if there exist some constants a, b, c such that

$$U \sim K(a, b, c) \;\; \text{and} \;\; V \sim \gamma(b, c).$$

If this condition holds, then $U' \sim \mathrm{Beta}(a, b)$ and $V' \sim K(a + b, -b, c)$.

Remark : This result has been proved earlier by K. and Vallois (2012) but under smothness assumptions on densities. Piliszek and Wesołowski (2018) used a nice change of measure technique to prove it without these assumptions.

3. A few words on Stein's method

- The distance between the distributions of two random variables W and Z can be quantified via

$$d_{\mathcal{H}}(W, Z) = \sup_{h \in \mathcal{H}} |\mathbb{E}(h(W) - \mathbb{E}(h(Z)|$$

  for a suitable class $\mathcal{H}$ of functions. Depending on $\mathcal{H}$ we have the Kolmogorov distance, the total variation distance or the Wasserstein distance.
- Stein (1974) derived a technique to obtain bounds for normal approximation.
- This technique relies on the fact that a rv W follows the $\mathrm{N}(0, \sigma^2)$ distribution if and only if

$$\mathbb{E}(\sigma^2 f'(W) - Wf(W)) = 0$$

  for regular functions f.

- This leads to the so-called Stein equation

$$\sigma^2 f'(x) - xf(x) = h(x) - \mathbb{E}(h(Z))$$

for $Z \sim N(0, \sigma^2)$.

- Given h, if $f = f_h$ is a solution of the Stein equation, then we have

$$|\mathbb{E}(h(W) - \mathbb{E}(h(Z)| = |\mathbb{E}(\sigma^2 f'(W) - Wf(W))|.$$

- Thus, to find a bound for $|\mathbb{E}(h(W)) - \mathbb{E}(h(Z))|$ for a rv W, it is enough to bound $|\mathbb{E}(\sigma^2 f'(W) - Wf(W))|$ for f solution of the above Stein equation.

- The method has been extended to several other distributions over the years.
- In this talk we are interested in deriving a bound to the solution of the Stein equation related to the generalized inverse Gaussian (GIG) and Kummer distributions.

4. Stein characterization in Döbler's framework

Consider a probability density g on $(0, \infty)$ verifying the following condition:

Assumption A: g is positive, differentiable on $(0, \infty)$ and there exist differentiable functions s and τ on $(0, \infty)$, such that s is positive, $\lim_{x \to 0} s(x)g(x) = \lim_{x \to \infty} s(x)g(x) = 0$ and, for all $x > 0$,

$$(s(x)g(x))' = \tau(x)g(x).$$

The GIG and Kummer densities satisfy Assumption A with

$$s(x) = x^2 \quad \text{and} \quad \tau_{p,a,b}(x) = \frac{b}{2} + (p+1)x - \frac{a}{2}x^2$$

for the GIG(p, a, b) distribution and

$$s(x) = x(x+1), \quad \tau(x) = (1-b)x - cx(1+x) + a$$

for the Kummer distribution $\mathrm{K}(a, b, c)$.

This enables us to treat both distributions in the same framework, using the general theory developed by Döbler (2015) for distributions fulfilling Assumption A and some regularity conditions. An advantage of this approach is that the bounds derived are optimal for Lipschitz test functions.

We refer to this general framework as Döbler's framework, but such distributions have also been considered by Schoutens (2001) with s a polynomial at most 2 and τ a decreasing linear function.

By using for instance the Stein density approach of Ley and Swan 2013, on obtains the following Stein characterization of distributions with density g satisfying Assumption A:
A positive random variable X has density g if and only if for any differentiable function f such that
$\lim_{x \to 0} s(x)g(x)f(x) = \lim_{x \to \infty} s(x)g(x)f(x) = 0$,

$$\mathbb{E}\left[s(X)f'(X) + \tau(X)f(X)\right] = 0.$$

The related Stein equation is

$$s(x)f'(x) + \tau(x)f(x) = h(x) - \mathbb{E}h(W)$$

where W is a random variable with density g.
The Stein equation has solution

$$f_h(x) = \frac{1}{s(x)g(x)} \int_0^x g(t)\,[h(t) - \mathbb{E}h(W)]\,dt$$

$$= \frac{-1}{s(x)g(x)} \int_x^{+\infty} g(t)\,[h(t) - \mathbb{E}h(W)]\,dt.$$

Bounds of the solution for bounded test functions

Proposition

Let W be a random variable with density g satisfying Assumption A. Suppose τ is decreasing. Given a function h, let f_h be the solution of the Stein equation related to g. For any bounded function h,

- *(Döbler, 2015):* $|f_h(x)| \leq \dfrac{\|h - E(h(W))\|}{2s(q_{0.5})g(q_{0.5})}$ *where $q_{0.5}$ is the median of W.*
- *(Konzou & Koudou, 2020):*

$$||f_h|| \leq M \, ||h(.) - \mathbb{E}h(W)||$$

where

$$M = \max\left(\frac{1}{s(\alpha)g(\alpha)} \int_0^{\alpha} g(t)dt; \frac{1}{s(\alpha)g(\alpha)} \int_{\alpha}^{+\infty} g(t)dt\right)$$

and α is the unique zero of τ on $(0, \infty)$.

Bounds of the solution for Lipschitz test functions

Our results for GIG and Kummer Stein equation for Lipschitz test functions are built on the following (Döbler, 2015):

Proposition

(Döbler, 2015). Let W be a random variable with density g satisfying Assumption A, and distribution function F, such that $\mathbb{E}(W) < \infty$. Given a function h, let f_h be the solution of the g-Stein equation given above. For any Lipschitz test function h,

$$|f_h(x)| \leq \|h'\| \frac{F(x)\mathbb{E}(W) - \int_0^x yg(y)dy}{s(x)g(x)}$$

$$= \|h'\| \frac{\int_0^x (\mathbb{E}(W) - yg(y))\, dy}{s(x)g(x)}.$$

Deriving an explicit expression of the bound given by the above proposition is not always possible. We succeeded in carrying this out, which is not straightforward, for the GIG and Kummer distributions, through the following result :

Theorem

(Konzou-Koudou-Gneyou, 2020). Consider a density g *on* $(0, \infty)$ *satisfying Assumption A for some functions* s *and* τ*, such that* τ *is polynomial and has the expression* $\tau(x) = a_2 x^2 + a_1 x + a_0$ *with* $a_2 < 0$ *and* $a_0 > 0$*.*
Let f_h *be the solution of the g-Stein equation.*
For any Lipschitz continuous test function h*,*

$$\|f_h\| \leq \frac{\mathbb{E}(W)}{a_0} \|h'\|$$

where W *is a random variable with density* g*.*

The proof is based on the following proposition :

Proposition

The function $U : x \mapsto \dfrac{\int_0^x (\mathbb{E}(W) - t)g(t)dt}{s(x)g(x)}$ *is decreasing on* $(0, \infty)$ *and*

$$\lim_{x \to 0} U(x) = \frac{\mathbb{E}(W)}{a_0} < \infty,$$

$$U(x) \sim_{x \to \infty} \frac{-1}{a_2 x},$$

$$\lim_{x \to \infty} U(x) = 0.$$

PROOF : By de l'Hôpital's rule,

$$
\begin{aligned}
\lim_{x\to 0} U(x) &= \lim_{x\to 0} \frac{(\mathbb{E}(W)-x)g(x)}{(s(x)g(x))'} \\
&= \lim_{x\to 0} \frac{(\mathbb{E}(W)-x)g(x)}{\tau(x)g(x)} \\
&= \frac{\mathbb{E}(W)}{\tau(0)} \\
&= \frac{\mathbb{E}(W)}{a_0} < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to \infty} U(x) &= \lim_{x\to \infty} \frac{(\mathbb{E}(W)-x)g(x)}{(s(x)g(x))'} \\
&= \lim_{x\to \infty} \frac{(\mathbb{E}(W)-x)g(x)}{\tau(x)g(x)} \\
&= \lim_{x\to \infty} \frac{\mathbb{E}(W)-x}{a_2x^2+a_1x+a_0} \\
&= 0.
\end{aligned}
$$

Let $m := \mathbb{E}(W)$. We have, for all $x > 0$, since $(s(x)g(x))' = \tau(x)g(x)$,

$$\begin{aligned} U'(x) &= \frac{(m-x)s(x)g^2(x) - \tau(x)g(x)\int_0^x (m-t)g(t)\,dt}{(s(x)g(x))^2} \\ &= \frac{(m-x)s(x)g(x) - \tau(x)\int_0^x (m-t)g(t)\,dt}{s^2(x)g(x)}. \end{aligned}$$

Let us prove that $A(x)$ defined by

$$A(x) := (m-x)s(x)g(x) - \tau(x)\int_0^x (m-t)g(t)\,dt$$

is negative for all $x > 0$.
We have

$$s(x)g(x) = \int_0^x \tau(t)g(t)\,dt.$$

Thus

$$A(x) = \int_0^x \left[(m-x)\tau(t) - (m-t)\tau(x)\right] g(t)\,dt.$$

For any $x > 0$, let

$$B_x(t) := (m - x)\tau(t) - (m - t)\tau(x).$$

We have

$$\begin{aligned} B_x(t) &= (m - x)\tau(t) - (m - x + x - t)\tau(x) \\ &= (m - x)(\tau(t) - \tau(x)) + (t - x)\tau(x) \\ &= (x - t)\left[a_2(x - m)t - a_2 mx \ - a_1 m - a_0\right] \end{aligned}$$

Suppose $0 < x \leq m$. Define

$$C_x(t) := a_2(x-m)t - a_2 mx \ - a_1 m - a_0. \tag{5.1}$$

Then, $t \mapsto C_x(t)$ is increasing, so that, for all $t \in (0, x]$,

$$C_x(t) \leq C_x(x) = a_2 x^2 - 2a_2 mx \ - a_1 m - a_0.$$

We now observe that $a_2 x^2 - 2a_2 mx \ - a_1 m - a_0$, polynomial in x, has discriminant

$$\Delta = 4a_2 \left[a_2 \left(\mathbb{E}(W) \right)^2 + a_1 \mathbb{E}(W) + a_0 \right].$$

By applying the g-Stein characterization to the function $x \mapsto f(x) = 1$, we have :

$$\mathbb{E}(\tau(W)) = a_2\mathbb{E}\left(W^2\right) + a_1\mathbb{E}(W) + a_0 = 0,$$

which implies

$$a_1\mathbb{E}(W) + a_0 = -a_2\mathbb{E}(W^2),$$

so that

$$\begin{aligned}\Delta &= 4a_2\left[a_2(\mathbb{E}(W))^2 - a_2\mathbb{E}(W^2)\right] \\ &= -4a_2^2 Var(W) \\ &< 0.\end{aligned}$$

As a consequence, since $a_2 < 0$, we have $a_2x^2 - 2a_2mx \ - a_1m - a_0 < 0$ for all x. It follows that, if $0 < x \leq m$, then $C_x(t) < 0$ for all $t \in (0, x]$, and therefore

$$B_x(t) = (x - t)C_x(t) \leq 0$$

for all $t \in (0, x]$. Thus, for all $x \in (0, m]$,

$$A(x) = \int_0^x B_x(t)g(t)\,dt \leq 0.$$

Similar arguments hold for $x > m$. Thus U is decreasing, ans since it is continuous, its supremum on $(0, \infty)$ is $\frac{\mathbb{E}(W)}{a_0}$. $\square$

5. Application to the GIG distribution

Stein characterization of the GIG distribution

Proposition

A random X follows the $GIG(p, a, b)$ distribution if and only if, for any "regular" function f ,

$$\mathbb{E}\left[X^2 f'(X) + \left(\frac{b}{2} + (p+1)X - \frac{a}{2}X^2\right) f(X)\right] = 0.$$

The corresponding Stein equation is

$$x^2 f'(x) + \left(\frac{b}{2} + (p+1)x - \frac{a}{2}x^2\right) f(x) = h(x) - \mathbb{E}h(W)$$

where h is a bounded function and W a random variable following the GIG distribution with parameters p, a, b.

Bound of the solution of the GIG Stein equation

The solution of Stein's equation is given by

$$f_h(x) = \frac{1}{x^2 g_{p,a,b}(x)} \int_0^x g_{p,a,b}(t)\,[h(t) - \mathbb{E}h(W)]\,dt$$

(see also Gaunt, 2017) .

Theorem

(Konzou-Koudou-Gneyou, 2020).
Let $p \in \mathbb{R}$, $a > 0$, $b > 0$. Let f_h be the solution of the GIG Stein equation defined above. For any Lipschitz continuous test function h,

$$\|f_h\| \leq \frac{2}{\sqrt{ab}} \frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})} \|h'\| .$$

Plugging the expressions of $\mathbb{E}(W)$ and $\mathbb{E}(W^2)$ for $W \sim GIG(p, a, b)$ in the following equality (obtained in the proof above)

$$a_2 \mathbb{E}\left(W^2\right) + a_1 \mathbb{E}\left(W\right) + a_0 = 0,$$

we obtain

$$\frac{K_{p+2}(\sqrt{ab})}{K_p(\sqrt{ab})} = 1 + \frac{2(p+1)}{\sqrt{ab}} \frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}$$

which is equivalent to the following well-known relation for the modified Bessel function of the third kind:

$$K_{p+2}(x) = K_p(x) + \frac{2(p+1)}{x} K_{p+1}(x), \quad x > 0.$$

6. Application to the Kummer distribution

Stein characterization of Kummer distribution

Proposition

A random variable X follows the Kummer distribution with density $k_{a,b,c}$ if and only if, for all "regular" function f such that the expectation exists,

$$\mathbb{E}\left[X(1+X)f'(X) + [(1-b)X - cX(1+X) + a]\, f(X)\right] = 0.$$

The corresponding Stein equation is

$$x(x+1)f'(x) + [(1-b)x - cx(1+x) + a]\, f(x) = h(x) - \mathbb{E}h(W)$$

where h is a bounded function and W has K(a,b,c) distribution.

Bound of the solution of the Kummer Stein equation

The bounded solution of this equation is

$$f_h(x) = \frac{1}{x(1+x)k_{a,b,c}(x)} \int_0^x k_{a,b,c}(t)\,[h(t) - \mathbb{E}h(W)]\,dt$$

$$= \frac{-1}{x(1+x)k_{a,b,c}(x)} \int_x^{+\infty} k_{a,b,c}(t)\,[h(t) - \mathbb{E}h(W)]\,dt.$$

Theorem

(Konzou-Koudou-Gneyou, 2020).
Let f_h be the solution of the Kummer Stein equation defined above. For any Lipschitz continuous test function h,

$$\|f_h\| \leq \frac{\psi(a+1, 2-b; c)}{\psi(a, 1-b; c)} \left\|h'\right\|.$$

Plugging the expressions of $\mathbb{E}(W)$ and $\mathbb{E}(W^2)$ for $W \sim K(a,b,c)$ in the following equality (obtained in the proof above)

$$a_2 \mathbb{E}\left(W^2\right) + a_1 \mathbb{E}\left(W\right) + a_0 = 0,$$

we obtain the relation

$$\psi(a+2, b+2; c) = \frac{1}{c(a+1)} \left((b-c)\psi(a+1, b+1; c) + \psi(a, b; c)\right).$$

Potential application : GIG distribution as the law of a continued fraction.

Theorem

Letac-Seshadri (1983).

- *Let X and Y be two independent random variables such that $X > 0$ and $Y \sim \gamma(p, a/2)$ for $p, a > 0$. Then $X =_d \frac{1}{Y+X}$ if and only if $X \sim \mathrm{GIG}(-p, a, a)$.*
- *Let X, Y_1 and Y_2 be three independent random variables such that $X > 0$, $Y_1 \sim \gamma(p, b/2)$ and $Y_2 \sim \gamma(p, a/2)$ for $p, a, b > 0$. Then $X =_d \frac{1}{Y_1 + \frac{1}{Y_2+X}}$ if and only if $X \sim \mathrm{GIG}(-p, a, b)$.*

If $(Y_i)_{i\geq 1}$ is a sequence of independent random variables such that

$$\mathcal{L}(Y_{2i-1}) = \mathcal{L}(Y_1) = \gamma(\lambda, b/2) \ \textit{and} \ \mathcal{L}(Y_{2i}) = \mathcal{L}(Y_2) = \gamma(\lambda, a/2); \ \ i \geq 1,$$

then

$$\mathcal{L}\left(\cfrac{1}{Y_1 + \cfrac{1}{Y_2 + \cfrac{1}{Y_3 + \ddots}}}\right) = GIG(-\lambda, a, b).$$

Question : Use Stein's method to derive bounds for the distance between

$$\mathcal{L}\left(\frac{1}{Y_1 + \frac{1}{Y_2 + \frac{1}{\cdots + \frac{1}{Y_n}}}}\right)$$

and $\mathrm{GIG}(p, a, b)$.

References

Barndorff-Nielsen, O. E. and Halgreen, C. (1977). Infinite divisibility of the Hyperbolic and generalized inverse Gaussian distribution. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **38**, 309–312.

Bhattacharya, R. N. and Waymire, E. C. (1990). *Stochastic Processes with Applications.* New York: Wiley.

Chebana, F., El Adlouni, S. and Bobée, B. (2010). Mixed estimation methods for Halphen distributons with applications in extreme hydrologic events. *Stoch. Environ. Res. Risk Assess.*, **24**, 359–376.

Döbler, C (2015). Stein's method of exchangeable pairs for the beta distribution and generalizations, *Electron. J. Probab.* **20**, 34 pp.

Gaunt, R. E. (2017). A Stein characterization of the generalized hyperbolic distribution. *ESAIM. Proba. Stat.* **21**, 303–316.

- Good, I. J. (1953).The population frequencies of species and the estimation of population parameters, *Biometrika*, **40**, 237–260.
- Iyengar, S. and Liao, Q. (1997). Modeling neural activity using the generalized inverse Gaussian distribution. *Biol. Cybern.* , **77**, 289–295.
- Jorgensen B. (1982) *Statistical Properties of the Generalized Inverse Gaussian Distribution*. Springer-Verlag, Heidelberg.
- Konzou, E. and Koudou, E. (2020). About the Stein equation for generalized inverse Gaussian and Kummer distributions. *ESAIM. Proba. Stat.*
- Koudou, A. E. and Ley, C. (2014). Characterizations of GIG laws: a survey. Probab. Surv. **11**, 161–176.

Koudou, A. E. and Vallois, P. (2012). Some independence properties of the Matsumoto-Yor type. *Bernoulli* **18**(1), 119-136.

Konzou, E. , Koudou, E. and Gneyou, K. (2020). New bounds for the solution and derivatives of the Stein equation for the generalized inverse Gaussian and Kummer distributions. Submitted.

Letac, G. and Wesołowski, J. (2000). An independence property for the product of GIG and gamma laws. *Ann. Prob.* **28**, 1371–1383.

Ley, C. and Swan, I. (2013). Stein's density approach and information inequalities. *Electron. Commun. Probab.* **18**.

Madan, D., Roynette, B. and Yor, M. (2008). Unifying Black-Scholes type formulae which involve Brownian last passage times up to a finite horizon. *Prép. IECL*, 2008/28.

Matsumoto, H. and Yor, M. (2001). An analogue of Pitman's $2M - X$ theorem for exponential Wiener functional, Part II: the role of the generalized inverse Gaussian laws. *Nagoya Math. J.* **162**, 65–86.

Piliszek, A. and Wesołowski, J. (2018). Change of measure technique in characterizations of the gamma and Kummer distributions, *J. Math. Anal. Appl.*, **458**, 2, 967–979.

Schoutens, W. Orthogonal polynomials in Stein's method. *J. Math. Anal. Appl.***253**, 2, 515-531.

Vallois, P. (1991). La loi Gaussienne inverse généralisée comme premier ou dernier temps de passage de diffusion. *Bull. Sc. Math.*, 2[e] Série, **115**, 301–368.