Approximation of martingale couplings in the weak adapted topology and applications

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I. Introduction

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III. Stability of Weak Martingale Optimal Transport

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III. Stability of Weak Martingale Optimal Transport

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ measurable

Optimal Transport problem: find

$$\mathrm{OT}(\mu, \nu, c) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y)\, \pi(dx, dy).$$

where

$$\Pi(\mu, \nu) = \left\{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \pi(dx, \mathbb{R}^d) = \mu(dx) \text{ and } \pi(\mathbb{R}^d, dy) = \nu(dy)\right\}.$$

Under $\mathbb{P}^*$, $S_{T_1} \sim \mu$ and $S_{T_2} \sim \nu$ (Breeden and Litzenberger, 1978).

$$\mathrm{MOT}(\mu, \nu, c) \leq \mathbb{E}^*[c(S_{T_1}, S_{T_2})] \leq -\mathrm{MOT}(\mu, \nu, -c) \text{ where}$$

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$$\mathrm{MOT}(\mu,\nu,c) = \inf_{M \in \Pi^{\mathrm{M}}(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x,y)\, M(dx,dy).$$

$$\Pi^{\mathrm{M}}(\mu,\nu) = \left\{ M(dx,dy) = \mu(dx) M_x(dy) \in \Pi(\mu,\nu) : x = \int_{\mathbb{R}^d} y M_x(dy)\ \mu \text{ a.e.} \right\}.$$

- Introduced by Beiglböck, Henry-Labordère and Penkner (2013) in a discrete time setting;
- Introduced by Galichon, Henry-Labordère and Touzi (2014) in a continuous time setting;
- Beiglböck and Juillet (2016) : left and right-curtain mart. couplings
- Many contributions since : Acciaio, Alfonsi, Backhoff-Veraguas, Bayraktar, Beiglböck, Brückerhoff, Corbetta, Cox, De March, Galichon, Ghoussoub, Guo, Guyon, Henry-Labordère, Hobson, Huesmann, Juillet, Kim, Lim, Neufeld, Nutz, Obłój, Pagès, Pammer, Sester, Siorpaes, Stebegg, Tan, Touzi, ...

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$$\mathrm{MOT}(\mu,\nu) = \inf_{M \in \Pi^{\mathrm{M}}(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x,y)\, M(dx,dy).$$

Theorem (Strassen (1965))

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be with finite first moment. Then

$$\begin{aligned} \Pi^{\mathrm{M}}(\mu,\nu) \neq \emptyset &\iff \mu \leq_{cx} \nu \\ &\iff \forall f : \mathbb{R}^d \to \mathbb{R} \text{ convex}, \int_{\mathbb{R}^d} f(x)\,\mu(dx) \leq \int_{\mathbb{R}^d} f(y)\,\nu(dy). \end{aligned}$$

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When μ and ν are finitely supported, the MOT problem boils down to linear programming.

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Stability of the MOT problem, i.e. is $(\mu,\nu) \mapsto \mathrm{MOT}(\mu,\nu)$ continuous?

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Stability of the MOT problem, i.e. is $(\mu,\nu) \mapsto \mathrm{MOT}(\mu,\nu)$ continuous?

1) Allows to approximate μ and ν by finitely supported measures $\hat{\mu}$ and $\hat{\nu}$, and solve $\mathrm{MOT}(\hat{\mu},\hat{\nu}) \sim \mathrm{MOT}(\mu,\nu)$.
2) In practice, μ and ν are extrapolated from the market data. The data is by nature noisy, hence it makes no sense to solve the MOT problem if the stability does not hold.

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Let $\rho \in [1, +\infty)$, $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) := \{\eta \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^\rho \eta(dx) < \infty\}$

$$\mathcal{W}_\rho(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int |x - y|^\rho \pi(dx, dy) \right)^{1/\rho}.$$

Metricizes the topology of weak convergence + convergence of moment of order ρ : $\int_{\mathbb{R}^d} |x - x_0|^\rho \eta(dx)$ for some $x_0 \in \mathbb{R}^d$.

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In dimension $d=1$, the comotonous or Hoeffding-Fréchet coupling is optimal :

$$\mathcal{W}_\rho^\rho(\mu,\nu) = \int_0^1 |F_\mu^{-1}(u) - F_\nu^{-1}(u)|^\rho du$$

where $F_\eta^{-1}(u) = \inf\{x \in \mathbb{R} : \eta((-\infty,x]) \geq u\}$ is the quantile function of $\eta \in \mathcal{P}(\mathbb{R})$.

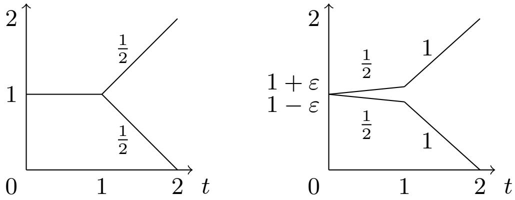
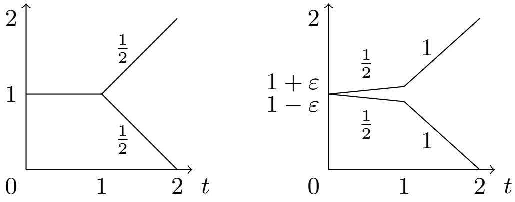


Figure: Two processes very close in Wasserstein distance but which yield radically unalike information. Example taken from Backhoff-Veraguas, Bartl, Beiglböck and Eder (2020).



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The adapted Wasserstein distance $\mathcal{AW}_\rho$ is defined for all couplings $\pi \in \Pi(\mu, \nu)$ and $\pi' \in \Pi(\mu', \nu')$ by

$$\mathcal{AW}_\rho(\pi, \pi') = \inf_{\chi \in \Pi(\mu, \mu')} \left(\int \left(|x - x'|^\rho + \mathcal{W}_\rho^\rho(\pi_x, \pi'_{x'}) \right) \chi(dx, dx') \right)^{1/\rho}$$

Approximation of couplings in the weak adapted topology

Let $\rho \in [1, +\infty)$, $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)$ and $(\mu^n)_{n\in\mathbb{N}}, (\nu^n)_{n\in\mathbb{N}} \in \mathcal{P}_\rho(\mathbb{R}^d)^{\mathbb{N}}$ respectively converge to μ and ν in $\mathcal{W}_\rho$.
It is well known that for each $\pi \in \Pi(\mu, \nu)$,

$$\inf_{\pi^n \in \Pi(\mu^n, \nu^n)} \mathcal{W}_\rho^\rho(\pi, \pi^n) \leq \mathcal{W}_\rho^\rho(\mu, \mu^n) + \mathcal{W}_\rho^\rho(\nu, \nu^n) \underset{n\to+\infty}{\longrightarrow} 0.$$

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Proposition

Let $\pi \in \Pi(\mu, \nu)$. Then

$$\inf_{\pi^n \in \Pi(\mu^n, \nu^n)} \mathcal{AW}_\rho(\pi, \pi^n) \underset{n\to+\infty}{\longrightarrow} 0.$$

If moreover $d = 1$ and for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ such that $\mu_n(\{x\}) > 0$, there exists $x' \in \mathbb{R}$ such that

$$\mu((-\infty, x')) \leq \mu^n((-\infty, x)) < \mu^n((-\infty, x]) \leq \mu((-\infty, x']),$$

which is for instance always satisfied when μ^n is non-atomic, then

$$\inf_{\pi^n \in \Pi(\mu^n, \nu^n)} \mathcal{AW}_\rho^\rho(\pi, \pi^n) \leq \mathcal{W}_\rho^\rho(\mu, \mu^n) + \mathcal{W}_\rho^\rho(\nu, \nu^n).$$

Remarks:

1) For all $n \in \mathbb{N}$ let $\pi^n \in \Pi(\mu^n, \nu^n)$ be such that $\mathcal{AW}_\rho(\pi, \pi^n) \underset{n\to+\infty}{\longrightarrow} 0$. If π is a martingale coupling, then, for $\chi^n \in \Pi(\mu^n, \mu)$,

$$\int_{\mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y\, \pi_x^n(dy) \right|^\rho \mu^n(dx)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y\, \pi_x^n(dy) \right|^\rho \chi^n(dx, dx')$$

$$\leq 2^{\rho-1} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x - x'|^\rho + \left| x' - \int_{\mathbb{R}^d} y\, \pi_x^n(dy) \right|^\rho \right) \chi^n(dx, dx')$$

$$= 2^{\rho-1} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x - x'|^\rho + \left| \int_{\mathbb{R}^d} y'\, \pi_{x'}(dy') - \int_{\mathbb{R}^d} y\, \pi_x^n(dy) \right|^\rho \right) \chi^n(dx, dx')$$

$$\leq 2^{\rho-1} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x - x'|^\rho + \mathcal{W}_1^\rho(\pi_x^n, \pi_{x'}) \right) \chi^n(dx, dx')$$

Hence, taking the infimum over $\chi^n \in \Pi(\mu^n, \mu)$,

$$\int_{\mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y\, \pi_x^n(dy) \right|^\rho \mu^n(dx) \leq 2^{\rho-1} \mathcal{AW}_\rho^\rho(\pi, \pi^n) \underset{n\to+\infty}{\longrightarrow} 0.$$

In that sense, $(\pi^n)_{n\in\mathbb{N}}$ is a sequence of almost martingale couplings.

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$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y\, \pi^n_x(dy) \right|^\rho \mu^n(dx) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y\, \pi^n_x(dy) \right|^\rho \chi^n(dx, dx') \\
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Hence, taking the infimum over $\chi^n \in \Pi(\mu^n, \mu)$,

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In that sense, $(\pi^n)_{n\in\mathbb{N}}$ is a sequence of almost martingale couplings.

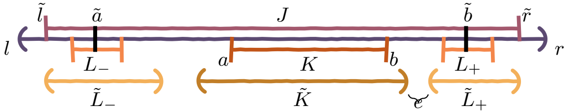
2) The convergence $\inf_{\pi^n\in\Pi(\mu^n,\nu^n)} \mathcal{AW}_\rho(\pi, \pi^n) \underset{n\to+\infty}{\longrightarrow} 0$ also holds when μ and ν are measures on Polish spaces.

Theorem

Let $\rho \in [1, +\infty)$, $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$ be such that $\mu \leq_{cx} \nu$ and $(\mu^n)_{n\in\mathbb{N}}, (\nu^n)_{n\in\mathbb{N}} \in \mathcal{P}_\rho(\mathbb{R})^{\mathbb{N}}$ be in convex order and respectively converge to μ and ν in $\mathcal{W}_\rho$. Then for all $M \in \Pi^{\mathrm{M}}(\mu, \nu)$,

$$\inf_{M^n \in \Pi^{\mathrm{M}}(\mu^n, \nu^n)} \mathcal{AW}_\rho(M, M^n) \underset{n\to+\infty}{\longrightarrow} 0.$$

Difficulty to restaure the martingale constraint which is global



Let

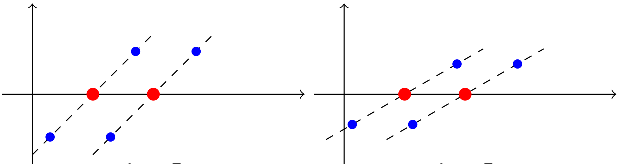
- $\rho \in [1, +\infty)$, $\mu, \nu, \mu^n, \nu^n \in \mathcal{P}_\rho(\mathbb{R})$, $\mu \leq_{cx} \nu$, $\mu^n \leq_{cx} \nu^n$;
- $\mu^n \to \mu$ and $\nu^n \to \nu$ in $\mathcal{W}_\rho$ as $n \to +\infty$;
- $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy for some $K < \infty$ the growth constraint

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R}, \ |c(x, y)| \leq K \left(1 + |x|^\rho + |y|^\rho\right).$$

Theorem (Backhoff-Veraguas and Pammer (2019) and independently Wiesel (2019))

There holds

$$\mathrm{MOT}(\mu^n, \nu^n) \underset{n \to +\infty}{\longrightarrow} \mathrm{MOT}(\mu, \nu).$$



Let $\mu = \frac{1}{2}\left(\delta_{(1,0)} + \delta_{(2,0)}\right)$ and for $\theta \in [0,\pi)$, $\nu_\theta(dy) = \int_{\mathbb{R}^2} \mu(dx) P_\theta(x,dy)$ where $P_\theta(x,dy) = \frac{1}{2}\left(\delta_{x+(\cos(\theta),\sin(\theta))}(dy) + \delta_{x-(\cos(\theta),\sin(\theta))}(dy)\right)$.
When $\theta \neq 0$, $\mu(dx)P_\theta(x,dy)$ is the only element of $\Pi^M(\mu,\nu_\theta)$ so that for $c(x,y) = |x-y|$,

$$\mathrm{MOT}(\mu,\nu_\theta) = \int_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|\mu(dx)P_\theta(x,dy) = 1.$$

For $\theta = 0$, $\Pi^M(\mu,\nu_0)$ also contains

$$\frac{1}{6}\delta_{((1,0),(0,0))} + \frac{1}{4}\delta_{((1,0),(1,0))} + \frac{1}{12}\delta_{((1,0),(3,0))} + \frac{1}{12}\delta_{((2,0),(0,0))} + \frac{1}{4}\delta_{((2,0),(2,0))} + \frac{1}{6}\delta_{((2,0),(3,0))}$$

so that

$$\mathrm{MOT}(\mu,\nu_0) \leq \frac{2}{3} < 1 = \lim_{\theta\to 0+} \mathrm{MOT}(\mu,\nu_\theta) \text{ while } \lim_{\theta\to 0+} \mathcal{W}_\rho(\nu_\theta,\nu) = 0.$$

WOT and WMOT problems

In dimension $d = 1$, we recover the stability result by proving the stability of more general problems.
Gozlan, Roberto, Samson and Tetali (2017) introduced the Weak Optimal Transport (WOT) problem: for two probability measures μ and ν defined on two Polish spaces E and F and a measurable map $C : E \times \mathcal{P}(F) \to [0, +\infty]$, find

$$\mathrm{WOT}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_E C(x, \pi_x)\, \mu(dx).$$

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If $E = F = \mathbb{R}^d$ and $\mu \leq_{cx} \nu$, the Weak Martingale Optimal Transport (WMOT) problem consists in the minimisation

$$\mathrm{WMOT}(\mu, \nu) = \inf_{M \in \Pi^{\mathrm{M}}(\mu,\nu)} \int_{\mathbb{R}^d} C(x, M_x)\, \mu(dx).$$

Example : $C(x, p) = -\sqrt{-\frac{2}{T_2 - T_1} \int_{(0,+\infty)} \ln\left(\frac{y}{x}\right)\, p(dy)}$ which gives the superreplication price of the VIX future according to Guyon, Menegaux and Nutz 2017

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$$C(x, p) = \int_{\mathbb{R}^d} c(x, y) p(dy) \text{ linear in the measure component.}$$

Theorem (Lower semicontinuity of WMOT)

Let $\rho \in [1, +\infty), C : \mathbb{R}^d \times \mathcal{P}_\rho(\mathbb{R}^d) \to \mathbb{R}$ *be convex in the second argument, lower semicontinuous satisfying the growth constraint*

$$\exists K < \infty,\ \forall (x,p) \in \mathbb{R}^d \times \mathcal{P}_\rho(\mathbb{R}^d),\ |C(x,p)| \leq K \left(1 + |x|^\rho + \int_{\mathbb{R}^d} |y|^\rho \, p(dy)\right).$$

Then

Stability of WMOT

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Then

(a) ***Existence and uniqueness:*** *for* $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)$ *with* $\mu \le_{cx} \nu$*, there exists* $M_\star \in \Pi^{\mathrm{M}}(\mu, \nu)$ *which minimises* $\mathrm{WMOT}(\mu, \nu)$ *and* $M_\star$ *is unique if* C *is strictly convex in its second argument.*

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(b) ***Lower semicontinuity:*** *for* $\mu^n, \nu^n \in \mathcal{P}_\rho(\mathbb{R}^d)$ *with* $\mu^n \le_{cx} \nu^n$ *and* $\lim_{n\to\infty} \left(\mathcal{W}_\rho(\mu_n,\mu) + \mathcal{W}_\rho(\nu_n,\nu)\right) = 0$, *then*

$$\mathrm{WMOT}(\mu,\nu) \le \liminf_{n\to+\infty} \mathrm{WMOT}(\mu^n,\nu^n).$$

Theorem (Lower semicontinuity of WMOT)

Let $\rho \in [1, +\infty), C : \mathbb{R}^d \times \mathcal{P}_\rho(\mathbb{R}^d) \to \mathbb{R}$ be convex in the second argument, lower semicontinuous satisfying the growth constraint

$$\exists K < \infty,\ \forall (x,p) \in \mathbb{R}^d \times \mathcal{P}_\rho(\mathbb{R}^d),\ |C(x,p)| \le K \left(1 + |x|^\rho + \int_{\mathbb{R}^d} |y|^\rho \, p(dy) \right).$$

Then

(a) ***Existence and uniqueness:*** *for $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)$ with $\mu \le_{cx} \nu$, there exists $M_\star \in \Pi^{\mathrm{M}}(\mu,\nu)$ which minimises $\mathrm{WMOT}(\mu,\nu)$ and $M_\star$ is unique if C is strictly convex in its second argument.*

(b) ***Lower semicontinuity:*** *for $\mu^n, \nu^n \in \mathcal{P}_\rho(\mathbb{R}^d)$ with $\mu^n \le_{cx} \nu^n$ and $\lim_{n\to\infty} (\mathcal{W}_\rho(\mu_n,\mu) + \mathcal{W}_\rho(\nu_n,\nu)) = 0$, then*

$$\mathrm{WMOT}(\mu,\nu) \le \liminf_{n\to+\infty} \mathrm{WMOT}(\mu^n,\nu^n).$$

(c) ***Convergence of optimizers:*** *If $\mathrm{WMOT}(\mu,\nu) = \lim_{n\to+\infty} \mathrm{WMOT}(\mu^n,\nu^n)$, then the accumulation points of $(M_\star^n)_n$ for $\mathcal{W}_\rho$ are minimizers of $\mathrm{WMOT}(\mu,\nu)$ and, when C is strictly convex in its second argument, then $\lim_{n\to\infty} \mathcal{AW}_\rho(M_\star^n, M_\star) = 0$.*

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- A sequence $M^n(dx, dy) = \mu(dx)M^n_x(dy)$ of elements of $\Pi^M(\mu, \nu)$ is generally not relatively compact for $\mathcal{AW}_\rho$.

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- For $P(dx,dp)=\mu(dx)P_x(dp)\in\mathcal{P}_\rho(\mathbb{R}^d\times\mathcal{P}_\rho(\mathbb{R}^d))$, by convexity of C in its second variable,

$$\int_{\mathbb{R}^d\times\mathcal{P}_\rho(\mathbb{R}^d)}C(x,p)P(dx,dp)=\int_{\mathbb{R}^d}\int_{\mathcal{P}_\rho(\mathbb{R}^d)}C(x,p)P_x(dp)\mu(dx)$$
$$\geq\int_{\mathbb{R}^d}C\left(x,\int_{\mathcal{P}_\rho(\mathbb{R}^d)}pP_x(dp)\right)\mu(dx).$$

Theorem (Stability of WMOT in dimension 1)

In dimension $d = 1$, suppose moreover that one of the following holds true:

(A) C is continuous.

(B) C is continuous in the second argument and $(\mu^n)_{n\in\mathbb{N}}$ converges strongly to μ (i.e. $\forall A \in \mathcal{B}(\mathbb{R})$, $\mu^n(A) \to \mu(A)$).

For $\mu^n \leq_{cx} \nu^n \in \mathcal{P}_\rho(\mathbb{R}^d)$ with $\lim_{n\to\infty} (\mathcal{W}_\rho(\mu_n, \mu) + \mathcal{W}_\rho(\nu_n, \nu)) = 0$,

(a) ***Stability:***

$$\lim_{n\to+\infty} \mathrm{WMOT}(\mu^n, \nu^n) = \mathrm{WMOT}(\mu, \nu).$$

(b) ***Convergence of optimizers:*** *The accumulation points of $(M_\star^n)_n$ for $\mathcal{W}_\rho$ are minimizers of $\mathrm{WMOT}(\mu, \nu)$ and, when C is strictly convex in its second argument, then $\lim_{n\to\infty} \mathcal{AW}_\rho(M_\star^n, M_\star) = 0$.*

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Let $M_\star$ be a minimiser of $\mathrm{WMOT}(\mu, \nu)$. According to our approximation result, there exist $M^n \in \Pi^{\mathrm{M}}(\mu^n, \nu^n)$, $n \in \mathbb{N}$ which converge in $\mathcal{AW}_\rho$ to $M_\star$.

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□

Thank you for your attention.