



Brenier-Schrödinger problem - A probabilistic approach of Navier-Stokes equations

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- 1 From fluid evolution to Brenier-Schrödinger**
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Euler equations - Newton principle :

$$\begin{cases} \partial_t v + \nabla_v v + \nabla p = 0, \\ \operatorname{div}(v) = 0, \\ v(0, \cdot) = v_0, \end{cases}$$

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- volume preserving flow $q \in G_{\text{vol}}$
- endpoint prescription q_1

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Arnold minimisation problem - least action principle ('66) :

$$\int_{[0,1] \times M} |\partial_t q_t(x)|^2 dt dx \rightarrow \min; [q_t \in G_{\text{vol}}, \forall 0 \leq t \leq 1], q_0 = \text{id}, q_1 = h$$

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- paths space $\Omega = \mathcal{C}^0([0, 1], \mathbb{R}^n)$, $X_t(\omega) = \omega_t$
- $Q \in \mathcal{P}(\Omega)$ path measure
- volume preserving measure $Q_t = Q(X_t \in \cdot) = \text{vol}$
- endpoint coupling $Q((X_0, X_1) \in \cdot) = \pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$

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Brenier's relaxation - minimisation on path measures ('89):

$$\mathbb{E}_Q \left[\int_0^1 |\dot{X}_t|^2 dt \right] \rightarrow \min; Q \in \mathcal{P}(\Omega), [Q_t = \text{vol}, \forall 0 \leq t \leq 1], Q_{01} = \pi,$$

Framework :

- M compact manifold with boundary
- ν : inward-pointing normal vector field

Navier-Stokes Equations :

$$\left\{ \begin{array}{l} \partial_t v + \nabla_v v - a \square v + \nabla p = 0, \\ \operatorname{div}(v) = 0, \\ \langle v, \nu \rangle = 0, \\ v(0, \cdot) = v_0. \end{array} \right. \quad (\text{on } \partial M)$$

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Hodge - de Rahm Laplacian \square : suggests Brownian processes.

$$\int_0^1 |\dot{X}_t|^2 dt = +\infty \quad \text{a.s.}$$

Notion of stochastic velocity?

On the restricted set of laws of Brownian paths with drift :

$$dX_t = \sqrt{2} dB_t + v_t dt,$$

the drift v gives a notion of stochastic velocity.

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- $\mathcal{E}(Q) = \mathbb{E}_Q \left[\int_0^1 |v_t|^p dt \right] \rightarrow$ porous-media equation
- $\mathcal{E}(Q) = \mathbb{E}_Q \left[\int_0^1 |v_t|^2 dt \right] \rightarrow$ Navier-Stokes equation
- $\mathcal{E}(Q) = \mathbb{E}_Q \left[\int_0^1 |v_t|_{H^1}^2 dt \right] \rightarrow$ Camassa-Holm equation

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Stochastic kinetic energy \Leftrightarrow relative entropy (Föllmer formula)

Brenier-Schrödinger problem

- $R \in \mathcal{P}(\Omega)$ law of the reflected Brownian motion
- relative entropy

$$H(P|R) = \int_{\Omega} \log \left(\frac{dP}{dR} \right) dP$$

- $(\mu_t)_{t \in \mathcal{T}}$ in $\mathcal{P}(M)$ for $\mathcal{T} \subset [0, 1]$
- $\pi \in \mathcal{P}(M^2)$

$$H(Q|R) \rightarrow \min, Q \in \mathcal{P}(\Omega), [Q_t = \mu_t, \forall t \in \mathcal{T}], Q_{01} = \pi. \quad (\text{BS})$$

Questions

- Link between Brenier-Schrödinger problem and kinetic energy.
- Link with Navier-Stokes equation for Nelson velocity (especially at the boundary)
- Existence of solutions.

Previous results : (Arnaudon et al. 2020) for $M = \mathbb{R}^n$ or $M = \mathbb{T}^n$.

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$$\begin{cases} dX_t = d_m^R X_t + \nu(X_t) dL_t(X) \\ d[X, X]_t = \text{Tr}(\cdot) dt \end{cases} \quad R\text{-a.s}$$

L : local time at ∂M

$d_m^R X$: martingale part of Ito differential

$$\begin{cases} dX_t = d_m^R X_t + v(X_t) dL_t(X) \\ d[X, X]_t = \text{Tr}(\cdot) dt \end{cases} \quad R\text{-a.s}$$

L : local time at ∂M

$d_m^R X$: martingale part of Ito differential

Theorem (Girsanov theorem)

Let P be such that $H(P|R) < \infty$. Then, P is the law of a semi-martingale and there exists $\zeta \in \mathcal{H}(P)$ such that P -a.s,

$$dX_t = d_m^P X_t + \zeta dt + v_{X_t} dL_t(X).$$

$\mathcal{H}(P)$: $g : [0, 1] \times M \rightarrow T^*M$ such that $g_t(\omega) \in T_{\omega_t}^*M$, adapted and

$$\mathbb{E}_P \left[\int_0^1 |g_t|^2 dt \right] < +\infty.$$

Density in terms of velocity

With the previous notations, the density of P is given by

$$\frac{dP}{dR} = \mathbb{1}_{\left\{\frac{dP}{dR} > 0\right\}} \exp \left(\int_0^1 \langle \zeta_t, d_m^R X_t \rangle - \frac{1}{2} \int_0^1 |\zeta_s|^2 ds \right).$$

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Entropy versus stochastic kinetic energy

$$H(P|R) = H(P_0|R_0) + \frac{1}{2} \mathbb{E}_P \left[\int_0^1 |\zeta_t|^2 dt \right]$$

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Minimal entropy = Minimal stochastic kinetic energy

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$$\mathcal{T} = \mathcal{T}_r \cup \mathcal{S}$$

\mathcal{T}_r : finite union of open intervals in $[0, 1]$

\mathcal{S} : finite subset of $(0, 1)$ such that $\mathcal{T} \cap \mathcal{S} = \emptyset$.

Regular solutions

$$P = \exp \left(\eta(X_0, X_1) + \sum_{s \in \mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_t(X_t) dt \right) R,$$

such that "well-defined quantities and sufficient regularity".

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such that "well-defined quantities and sufficient regularity".

- Formula comes from primal/dual analysis of the problem
- To a data of \mathcal{T} , (μ_t) and π corresponds η , (θ_s) and p .

Theorem (García-Zelada, H., 2021)

For P_1 -almost all $y \in M$, the backward stochastic velocity $\overset{\leftarrow}{v}^y$ exists and derives from a potential φ^y :

$$\overset{\leftarrow}{v}_t^y(X) = -\nabla \varphi_t^y(X_t), P\text{-a.s.} \quad (1)$$

Moreover, it satisfies :

$$\left\{ \begin{array}{ll} \left(\partial_t + \nabla_{\overset{\leftarrow}{v}} \right) \overset{\leftarrow}{v} = \frac{1}{2} \square \overset{\leftarrow}{v} - \mathbf{1}_{\mathcal{T}}(t) \nabla p, & 0 \leq t < 1, t \notin \mathcal{S}, z \in M, \\ \overset{\leftarrow}{v}_t - \overset{\leftarrow}{v}_{t-} = \theta_t(\cdot), & t \in \mathcal{S}, z \in M, \\ \langle \overset{\leftarrow}{v}, \nu(z) \rangle = 0, & z \in \partial M, \\ \overset{\leftarrow}{v}_0 = -\nabla \eta(\cdot, y), & z \in M. \end{array} \right. \quad (2)$$

Few comments :

- backward velocity :

$$\overset{\leftarrow}{v}_t^P = \lim_{h \rightarrow 0^+} \frac{1}{h} \mathbb{E}_P \left[\overrightarrow{X_{t-h \wedge \tau_t} X_t} \mid X_{[t,1]} \right]$$

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Theorem (García-Zelada, H., 2021)

The current velocity v_{cu}^P satisfies the continuity equation :

$$\partial_t \mu_t + \operatorname{div}(\mu_t v_{cu}) = 0, \forall t \in \mathcal{T}.$$

(ideas of proof)

Step 1. Equation for the forward potential ψ .

$$\frac{dP_{[0,t]}^x}{dR_{[0,t]}^x} = \exp \left(\sum_{s \in \mathcal{S}, s \leq t} \theta_s(X_s) + \int_{\mathcal{T} \cap [0,t]} p_r(X_r) dr + \psi_t^x(X_t) \right), R^x\text{-a.s.}$$

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$$\frac{dP_{[0,t]}^x}{dR_{[0,t]}^x} = \exp \left(\int_0^t \langle \zeta_s^x, d_m^{R^x} X_s \rangle - \frac{1}{2} \int_0^t |\zeta_s^x|^2 ds \right), P^x\text{-a.s.}$$

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$$\langle \zeta_t^x, d_m^{R^x} X_t \rangle - \frac{1}{2} |\zeta_t^x|^2 dt = \mathbb{1}_{\mathcal{S}}(t) \theta_t + p_t dt + d\psi_t^x(X_t), P^x\text{-a.s.}$$

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$$\overleftarrow{v}^x = \zeta_t^x = \nabla \psi_t^x(X_t), dt \otimes P^x\text{-almost surely.}$$

(ideas of proof)

$$\overset{x}{v} = \zeta_t^x = \nabla \psi_t^x(X_t), dt \otimes P^x\text{-almost surely.}$$

$$\begin{cases} \partial_t \psi_t^x + \frac{1}{2} \Delta \psi_t^x + \frac{1}{2} \|\nabla \psi_t^x\|^2 + \mathbb{1}_{\mathcal{T}}(t) p_t = 0, & t \in [0, 1) \setminus \mathcal{S}, \\ \psi_t^x - \psi_{t^-}^x = -\theta_t, & t \in \mathcal{S}, \\ \langle \nabla \psi_t^x(z), \nu_z \rangle = 0, & z \in \partial M, \\ \psi_1^x = \eta(x, \cdot), & t = 1, \end{cases}$$

(ideas of proof)

Step 2. Forward velocity's "fluid equations".

$$\left\{ \begin{array}{ll} \left(\partial_t + \nabla_{\vec{x}} \right) \vec{v} = -\frac{1}{2} \square \vec{v} - \mathbb{1}_{\mathcal{T}}(t) \nabla p, & t \in [0, 1) \setminus S, \\ \vec{v}_t - \vec{v}_{t^-} = -\nabla \theta_t, & t \in S, \\ \langle \vec{v}(z), \nu_z \rangle = 0, & z \in \partial M, \\ \vec{v}_1 = \nabla(\eta(\cdot, x)), & t = 1. \end{array} \right.$$

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Step 3. Time reverse.

$$\overset{\leftarrow}{v}_t = - \overset{y}{\underset{v}{\rightarrow}}_{1-t}^{P^*} \circ \text{rev}$$

where $\text{rev}(\omega)_t = \omega_{1-t}$ and $P^* = \text{rev}_* P$.

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- Homogeneous spaces
- Quotient spaces
- Gaussian problem

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$$H(Q|R) \rightarrow \min, Q \in \mathcal{P}(\Omega), [Q_t = \text{vol}, \forall t \in [0, 1]], Q_{01} = \pi. \quad (\text{iBS})$$

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Method : If there exists a path measure Q such that $H(Q|R) < \infty$, $Q_t = \mu_t, \forall t \in \mathcal{T}$ and $Q_{01} = \pi$, then there exists a unique solution.

$$H(Q|R) \rightarrow \min, Q \in \mathcal{P}(\Omega), [Q_t = \text{vol}, \forall t \in [0, 1]], Q_{01} = \pi. \quad (\text{iBS})$$

Method : If there exists a path measure Q such that $H(Q|R) < \infty$, $Q_t = \mu_t, \forall t \in \mathcal{T}$ and $Q_{01} = \pi$, then there exists a unique solution.

Previous results : (Arnaudon et al., 2020) $M = \mathbb{T}^n$ with the necessary and sufficient condition $H(\pi|R_{01}) < \infty$.

(M, g) compact Riemannian manifold. G group of isometries. $G \curvearrowright M$ transitive.

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Examples : S^n, \mathbb{T}^n .

Theorem (García-Zelada, H., 2021)

(iBS) admits a unique solution if and only if $H(\pi|R_{01}) < \infty$.

(idea of the proof)

$$Q = \int_{M^3} R(\cdot | X_0 = x, X_{1/2} = z, X_1 = y) \sigma(dx dz dy),$$

where $\sigma(dx dz dy) = \pi(dx dy) \text{vol}(dz)$

(idea of the proof)

$$Q = \int_{M^3} R(\cdot | X_0 = x, X_{1/2} = z, X_1 = y) \sigma(dx dz dy),$$

where $\sigma(dx dz dy) = \pi(dx dy) \text{vol}(dz)$

- $Q_{01} = \pi$
- $H(Q|R) < \infty$. Compactness argument +

$$H(Q|R) = H(\pi|R_{01}) + \int_{M^2} H(\text{vol}|R_{1/2}^{xy}) \pi(dx dy).$$

- $Q_t = \text{vol}$. Uniqueness of invariant measure

$N = M/G$ with M Riemannian manifold and G reflections group.

$q : M \rightarrow N$, quotient, transports :

- volume measure
- Brownian motion to reflected Brownian motion
- Path measures satisfying marginals and entropy conditions

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 $q : M \rightarrow N$, quotient, transports :

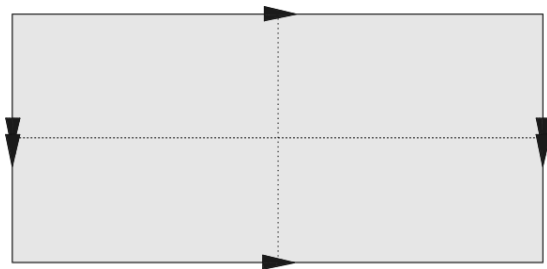
- volume measure
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- Path measures satisfying marginals and entropy conditions

Theorem (García-Zelada, H., 2021)

If $(iBS)_{M,\pi}$ admits a solution, then $(iBS)_{N,(q \times q)_\pi}$ admits a solution.
 In particular, if M is a compact homogeneous space, then $(iBS)_{N,\tilde{\pi}}$ admits a solution for every $\tilde{\pi} \in \mathcal{P}(N^2)$ with vol_N marginals and $H(\pi|\tilde{R}_{01}) < \infty$.*

Example - Hyper-rectangle

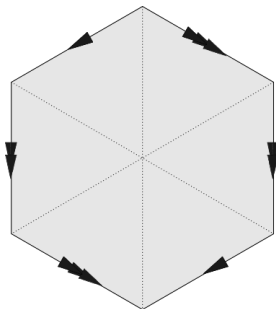
An n -dimensional rectangular box can be seen as quotient of a torus \mathbb{T}^n .



→ Existence and uniqueness of solution if and only if finite entropy.

Example - Equilateral triangle

An equilateral triangle can be seen as quotient space of the torus \mathbb{T}^2 .



→ Existence and uniqueness of solution if and only if finite entropy.

Minimisation on path measure on \mathbb{R}^n . R Brownian motion in \mathbb{R}^n .

$$H(P|R) \rightarrow \min; [P_t = \mathcal{N}(0, 1/4 \text{id}), \forall 0 \leq t \leq 1], P_{01} = \pi \quad (BS_\gamma)$$

Theorem (García-Zelada, H., 2021)

(BS_γ) admits a unique solution if and only if $H(\pi|R_{01}) < \infty$.

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- Generalisation of this probabilistic approach
 - to other equations ("to a kinetic energy, a fluid equation")
 - On different spaces

Thank you for your attention.