On population growth with catastrophes

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- With α_1, $a > 0$, consider the growth dynamics

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for some growth field $\alpha(x) = \alpha_1 x^a$. Integrating when $a \neq 1$, we get formally

$$x_t(x) = \left(x^{1-a} + \alpha_1 (1-a) t\right)^{1/(1-a)}. \tag{2}$$

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   - The role of 0 and of $+\infty$
   - Piecewise deterministic Markov process (PDMP)
   - First jump distribution
   - Classification of state 0
2. Speed measure and Harris recurrence
3. Expected return times to 0

The role of 0 and of $+\infty$

In general, α being positive on $(0, \infty)$, we have

$$\int_x^{x_t(x)} \frac{dy}{\alpha(y)} = t.$$

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$I_0(x) < \infty \iff$ state 0 is reflecting, $I_\infty(x) < \infty \iff$ state ∞ is accessible,

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$$\begin{aligned} I_0(x) &< \infty \Longleftrightarrow \text{state } 0 \text{ is reflecting, } I_\infty(x) < \infty \Longleftrightarrow \text{state } \infty \text{ is accessible,} \\ I_0(x) &= \infty \Longleftrightarrow \text{state } 0 \text{ is absorbing, } I_\infty(x) = \infty \Longleftrightarrow \text{state } \infty \text{ is inaccessible.} \end{aligned}$$

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In the separable case, $H\left(x, y\right) = \frac{h(y)}{h(x)}$ where h is non decreasing function.

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$$dX_t = \alpha(X_{t-})\, dt - \Delta(X_{t-}) \int_0^{\infty} 1_{\{z \leq \beta(X_{t-})\}} M(dt, dz), \; X_0 = x \geq 0 \qquad (3)$$

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- Notice that, between successive jumps the only possibility for the process to go down is by jumping.

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- $h(x) = x$ in which case $H(x, 0) = 0$.

Q: What is the law of the first jump?

First jump distribution in case of $I_\infty(x) = \infty$

Defining

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- We suppose that $I_\infty(x) = \infty$
- $X_t = x_t(x)$ on $t < T_x$, we have

$$\mathrm{P}\left(T_x > t\right) = \mathrm{P}_x\left(\int_0^t \int_0^\infty 1_{\{z \leq \beta(x_s(x))\}} M\left(ds, dz\right) = 0\right).$$

First jump distribution in case of $I_\infty(x) = \infty$

- $\gamma(x) := \beta(x)/\alpha(x)$ and $\Gamma(x) := \int^x \gamma(y)\,dy$, we get, since $\alpha > 0$ on $(0, \infty)$,

$$\mathrm{P}\left(T_x > t\right) = e^{-\int_0^t \beta(x_s(x))ds} = e^{-[\Gamma(x_t(x)) - \Gamma(x)]}, \text{ for all } t \geq 0. \tag{5}$$

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Assumption 2

$\Gamma(0) > -\infty$.

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Classification of state 0

Notice that for all $x > 0$,

$$l_0(x) = \int_0^x \frac{dy}{\alpha(y)}.$$

- $H(x,0) > 0$ and $l_0(x) < \infty$: regular (accessible and reflecting).
- $H(x,0) > 0$ and $l_0(x) = \infty$: exit (accessible and absorbing).
- $H(x,0) = 0$ and $l_0(x) < \infty$: entrance (inaccessible and reflecting).
- $H(x,0) = 0$ and $l_0(x) = \infty$: natural (inaccessible and absorbing).

Speed measure

The associated infinitesimal generator is given for any smooth test function u by

$$Gu(x) = \alpha(x)u'(x) + \beta(x)\int_0^x [u(y) - u(x)]H(x, dy), x \geq 0. \quad (6)$$

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The explicit expression of the invariant measure π in the separable case $H(x, y) = \frac{h(y)}{h(x)}$ is given by :

$$\pi(y) = C\frac{h(y)}{\alpha(y)}e^{-\Gamma(y)}, \quad (7)$$

up to a multiplicative constant $C > 0$.

Recurrence of X

Definition 2 (J. Azéma, M. Duflo, D. Revuz)

X is called *Harris recurrent* if there exists some σ-finite measure m on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that for all $A \in \mathcal{B}(\mathbb{R}_+)$,

$$m(A) > 0 \text{ implies } P_x\left(\int_0^\infty 1_A(X_s)ds = \infty\right) = 1 \text{ for all } x \in \mathbb{R}_+.$$

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X is then called *positive recurrent* (or also sometimes *ergodic*) if $\pi(\mathbb{R}_+) < \infty$, *null recurrent* if $\pi(\mathbb{R}_+) = \infty$.

Example 3

If $h(x) \sim e^{\Gamma(x)}$ as $x \to \infty$, we have $\pi(x) \sim \frac{1}{\alpha(x)}$, as $x \to \infty$. In particular, $\int^{\infty} \pi(y)dy < \infty$ if and only if $I_{\infty}(x) < \infty$ for some (and thus all) $x > 0$.

Exit probabilities and excursions

Throughout this section we impose Assumptions 1 and 2. With $x > 0$, we introduce

$$\tau_{x,0} = \inf\{t > 0 : X_t = 0 \mid X_0 = x\}$$

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$$p\left(x,b\right) = h(0)(1 - p(0,b)) \int_x^b \frac{\gamma\left(y\right)}{h\left(y\right)} e^{\Gamma(y)} dy.$$

where $p(b,b) = 0$.

Exit probabilities and excursions

Let

$$s(x) = \int_0^x \frac{\gamma(y)}{h(y)} e^{\Gamma(y)} dy, \Gamma(y) = \int_0^y \gamma(t) dt. \tag{8}$$

Notice that under Assumption 2 and supposing that $h(0) > 0$, $s(x)$ is well-defined for any $x \geq 0$.
We obtain

$$p(0,b) = \frac{h(0)s(b)}{1 + h(0)s(b)} \text{ and } \mathrm{P}\left(\tau_{x,0} < \tau_{x,b}\right) = p(0,b)[1 - \frac{s(x)}{s(b)}]. \tag{9}$$

Proposition 1

Grant Assumptions 1 and 2 and suppose moreover that $H(x,y) = \frac{h(y)}{h(x)}$ *with* $h(0) > 0$, *that* $I_\infty(x) = \infty$ *and* $I_0(x) < \infty$.

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Then the process is recurrent if and only if $s(\infty) = \infty$, *where the function* $s(x)$ *is given by* (8).

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In this latter case, $\tau_{x,0} < \infty$ *almost surely, and the unique invariant measure possesses a Lebesgue density on* $\mathbb{R}_+$ *which is given by* (7). *The process is positive recurrent if* $\int^\infty \frac{h(x)}{\alpha(x)} e^{-\Gamma(x)} dx < \infty$, *null-recurrent else.*

Classification of the recurrence/ transience of state 0

Under Assumption 1 and 2, we have :

- $s(\infty) = \infty, I_0(x) < \infty$: 0 is recurrent, positive recurrent iff $\int^{\infty} \frac{h(x)}{\alpha(x)} e^{-\Gamma(x)} dx < \infty$.

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- $s(\infty) < \infty, I_\infty(x) = \infty$: The process is transient (converges to $+\infty$ with positive probability).
- $s(\infty) < \infty, I_\infty(x) < \infty$: The process is either transient (converges to $+\infty$ with positive probability) or hits state ∞ in finite time ($\tau_{x,\infty} < \infty$ with positive probability).

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-We suppose $0 < H\left(x, 0\right) < 1$ for all x. If $x > 0$, we have

$$\tau_{x,0} \overset{d}{=} T_x 1\left(X_{T_x} = 0\right) + 1\left(X_{T_x} > 0\right)\left(T_x + \tau'_{X_{T_x},0}\right), \tag{10}$$

where $\tau'_{X_{T_x}}$ is independent of $\mathcal{F}_{T_x}$ and distributed as $\tau_{X_{T_x}}$. This implies

$$u(x) = \mathrm{E}\left(\tau_{x,0}\right) = \mathrm{E}\left(T_x\right) + \int_{0^+}^{\infty} \mathrm{P}\left(X_{T_x} \in dy\right)\mathrm{E}\left(\tau_{y,0}\right), \; x > 0.$$

We pose $u(0+) := \lim_{x\to 0} u(x) = \mathrm{E}(\tau_{0,0})$.

Theorem 4

Let π the unique invariant measure given in (7), where the constant C is chosen such that π is tuned to a probability. Then for any Borel subset B of $\mathbb{R}_+$,

$$\pi(B) = \frac{1}{u(0+)} \mathrm{E}_0 \int_0^{\mathcal{T}_{0,0}} 1_B(X_s)ds. \tag{11}$$

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Suppose now moreover that $\pi(\beta) \in (0, \infty)$ and that $\alpha(0) > 0$, then

$$\mathsf{E}(\tau_{0,0}) = u(0+) = \frac{1}{Ch(0)}. \tag{12}$$

Proof.

- Applying (11) with $B = [0, \varepsilon]$, we obtain

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} \pi(y)dy = \frac{1}{u(0+)} E_0 \left(\frac{1}{\varepsilon} \int_0^{T_{0,0}} 1_{\{X_s \le \varepsilon\}} ds \right)$$

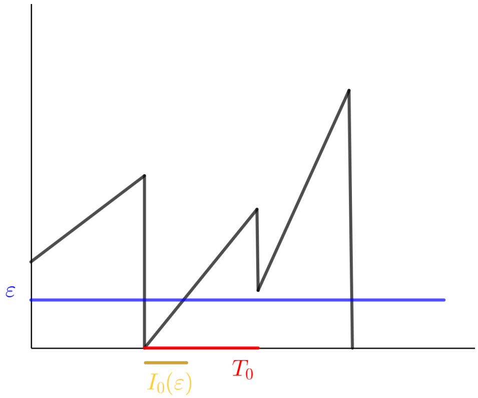
- $\varepsilon \to 0$, clearly the left hand side converges to $\pi(0) = C(h(0)/\alpha(0))e^{-\Gamma(0)} = Ch(0)/\alpha(0)$, since we have chosen $\Gamma(0) = 0$.
- We want to show that

$$\lim_{\varepsilon \to 0} E_0 \left(\frac{1}{\varepsilon} \int_0^{T_{0,0}} 1_{\{X_s \le \varepsilon\}} ds \right) = 1/\alpha(0).$$

□

Clearly,

$$E_0 \int_0^{T_{0,0}} 1_{\{X_s \le \varepsilon\}} ds = P(T_0 > I_0(\varepsilon)) \cdot I_0(\varepsilon) + R(\varepsilon), \tag{13}$$



We also have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathrm{P}(T_0 > l_0(\varepsilon)) \cdot l_0(\varepsilon) = \lim_{\varepsilon \to 0} e^{-\Gamma(\varepsilon)} \frac{1}{\varepsilon} \int_0^{\varepsilon} \frac{1}{\alpha(y)} dy = \frac{1}{\alpha(0)}$$

so it only remains to show that $R(\varepsilon) = o(\varepsilon)$.

Theorem 5

Grant the above assumptions together with $\pi(\beta) < \infty$, and suppose that $H(x, y) = h(y)/h(x)$, where h is differentiable, non-decreasing, with $h(0) > 0$ and $\alpha(0) > 0$. We choose $\Gamma(0) = 0$. Then $u(x)$ is given by

$$\begin{aligned} u(x) &= u(0) + \int_0^x dy \frac{\gamma(y)\, e^{\Gamma(y)}}{h(y)} \int_y^\infty e^{-\Gamma(z)} \frac{h(z)}{\alpha(z)} dz - \int_0^x \frac{1}{\alpha(y)} dy \\ &= u(0) + s(x) \int_x^\infty \pi(y) dy + \int_0^x s(y)\pi(y) dy - \int_0^x \frac{1}{\alpha(y)} dy, \end{aligned} \tag{14}$$

where $u(0)$ is given by (12).

References

- Jacques Azéma, Marie Duflo, and Daniel Revuz. "Mesure invariante des processusde Markov récurrents". In:Séminaire de Probabilités III Université de Strasbourg.Springer, 1969, pp. 24–33
- Goncalves, Branda and Huillet, Thierry and Löcherbach, Eva. "ON POPULATION GROWTH WITH CATASTROPHES". https://hal.archives-ouvertes.fr/hal-02900249 (2020)

Thank you for your attention