# Propagation of chaos and Poisson hypothesis for replica-mean-field networks

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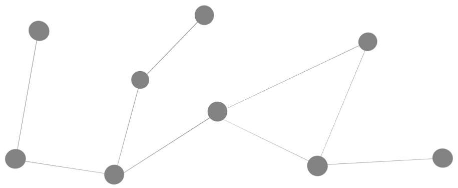
Joint works with F. Baccelli and T. Taillefumier

# Motivation

## Interacting particles

Interest: evolution in time of $N$ interacting particles. We see them as a network with $N$ nodes that send each other information.



## Examples of applications

- Statistical physics
- Epidemiology
- Communication networks
- Opinion dynamics
- Neuroscience (populations of neurons)

## Models of $N$ interacting particles

Network with $N$ nodes interacting through *point processes*. The edges are the support of interactions and point processes on the nodes mark the times at which the interactions happen. Usually: system of $N$ SDEs describing the evolution of the system.

## Advantages

- Versatile formalism;
- Close to reality.

## Problems

- High complexity that prevents mathematical solvability.
- Limitation to costly numerical simulations.

## Remark

SDE not PDE because of high variability in the dynamics of interest

# Replica mean fields

## Goal

We want to simplify the initial model to make it *tractable*. What simplifying assumptions should we take?

## "Classical" mean field

We look at the limit when the number of nodes goes to infinity and the weights of interactions between nodes go to 0.

## Main problem

The geometry of the initial model is not preserved. In particular, the correlations between particles dues to the finite size of the system are lost
→ problematic in certain applications (ex: neuroscience).

## One solution: replica mean-fields

- Given a network of $K$ interaction nodes, we consider $M$ identically distributed replicas (copies) of the initial network.
- If a node $i$ would interact with node $j$ in replica $m \in \{1, ..., M\}$, instead a replica is uniformly and independently chosen in $\{1, ..., M\} \setminus \{m\}$ (*routing* operation) and node $i$ interacts with node $j$ in the replica thus chosen.

## Goal

Show a convergence to a tractable model when the number $M$ of replicas goes to infinity.

# Replica mean field models in continuous time

## The base Galves-Löcherbach model

Collection of $K$ neurons whose spiking activities are given by the realization of a system of simple point processes without any common points $\mathbf{N} = \{N_i\}_{1 \le i \le K}$ on $\mathbb{R}^+$ defined on some measurable space $(\Omega, \mathcal{F})$. Their stochastic intensities $\lambda_1, \ldots, \lambda_K$ evolve according to

$$\lambda_i(t) = \lambda_i(0) + \frac{1}{\tau_i} \int_0^t (b_i - \lambda_i(s))\,\mathrm{d}s + \sum_{j \neq i} \mu_{i,j} \int_0^t N_j(\mathrm{d}s) + \int_0^t (r_i - \lambda_i(s)) N_i(\mathrm{d}s). \tag{1}$$

## Interpretation

Present = Past + Relaxation to base rate $b_i$+ Interactions + Reset to $r_i$

## The RMF dynamics

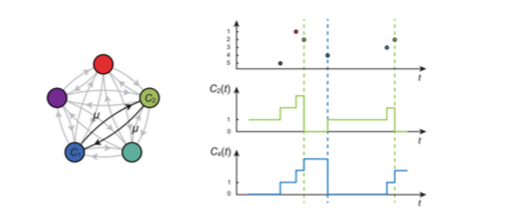
The dynamics of the $M$-replica system are characterized by

$$\begin{aligned}\lambda_{m,i}(t) = \lambda_{m,i}(0) + \frac{1}{\tau_i}\int_0^t (b_i - \lambda_{m,i}(s))\,\mathrm{d}s \\ + \sum_{n\neq m}\sum_{j\neq i}\mu_{i,j}\int_0^t \mathbb{1}_{\{V_{n,i,j}(s)=m\}}\, N_{n,j}(\mathrm{d}s) \\ + \int_0^t (r_i - \lambda_{m,i}(s))N_{m,i}(\mathrm{d}s),\end{aligned} \tag{2}$$

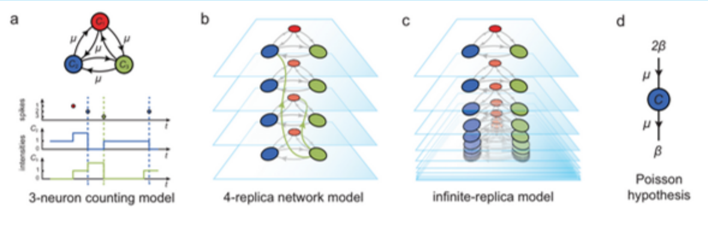
where $\{V_{m,i,j}(t)\}_{t\in\mathbb{R}}$ be cadlag stochastic processes such that for each spiking time $T$, i.e., each point of $N_{m,i}$, the random variables $\{V_{m,i,j}(T)\}_j$ are mutually independent, independent from the past and uniformly distributed on $\{1, ..., M\} \setminus \{m\}$, taking values in the set of integers $\mathcal{V}_{m,i} = \{v \in [1, \ldots, M]^K \,|v_i = m \text{ and } v_j \neq m, j \neq i\}$.

## Base toy GL model and associated RMF model

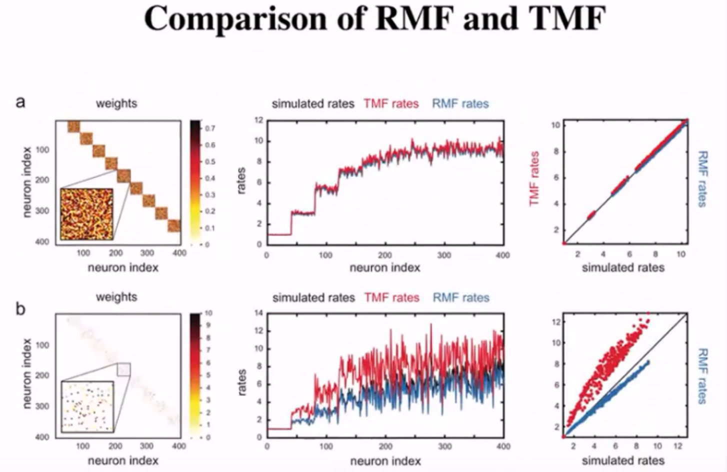
### Toy model




### Replica mean field



## Comparing RMF and classical "thermodynamic" mean field




Credit : F. Baccelli and T. Taillefumier

### The main goal

Prove the Poisson Hypothesis introduced by Kleinrock for the RMF dynamics: as $M \to \infty$, the replicas become independent (propagation of chaos) and the arrivals process to a given node in a given replica converges to some Poisson process.

## The limit process

For $1 \leq k \leq K$ we consider point processes $\tilde{N}_1, \ldots, \tilde{N}_K$ with respective stochastic intensities $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_K$ verifying the following stochastic differential equations:

$$\begin{aligned} \tilde{\lambda}_i(t) = \tilde{\lambda}_i(0) + \frac{1}{\tau_i} \int_0^t (b_i - \tilde{\lambda}_{m,i}(s)) \, \mathrm{d}s \\ + \sum_{j \neq i} \int_0^t M_{i,j}(\mathrm{d}s) + \int_0^t \left( r_i - \tilde{\lambda}_i(s) \right) \tilde{N}_i(\mathrm{d}s), \end{aligned} \tag{3}$$

where $M_{i,j}$ are Poisson point processes with intensity $m_{i,j}(s) = \mu_{i,j} \, \mathbf{E}[\tilde{\lambda}_j(s)], \mu_{i,j} \in \mathbb{R}^+$

### Main result (F.Baccelli and M.D.)

Let $T > 0$. Let $M \in \mathbb{N}$. Then for all $i \in \{1, ..., K\}$, for all $m \in \{1, \dots, M\}$, $\lambda_{m,i} \Rightarrow \tilde{\lambda}_i$ in $D([0, T])$ when $M \to \infty$.

### Idea of proof

- Represent the M-replica and limit processes through Poisson embeddings
- Couple the $M$-replica processes with i.i.d copies of the limit process through their initial conditions and Poisson embeddings

### Poisson embedding (Brémaud)

Let $N$ be a point process on $\mathbb{R}$ with the $\mathcal{F}_t$-predictable stochastic intensity $\{\mu(t)\}_{t\in\mathbb{R}}$. Then there exists a Poisson point process $\bar{N}$ with intensity 1 on $\mathbb{R}^2$ such that, for all $C \in \mathcal{B}(\mathbb{R}^2)$,

$$N(C) = \int_{C\times\mathbb{R}} \mathbb{1}_{[0,\mu(s)]}(u)\bar{N}(\mathrm{d}s \times \mathrm{d}u).$$

# Discrete-time replica mean fields

## Motivation

Study the RMF for a class of discrete time processes wide enough to cover different fields of application.

## Intuitive idea

- Discrete time evolution on a discrete state space.
- At each time step, each node can activate and be *fragmented* with a probability depending on its state.
- If fragmentation occurs, the node *interacts* with the other nodes of the network.
- Whether fragmentation occurs or not, the node receives the interactions of the other fragmented nodes and *aggregates* them to its state.

## Example: discrete time Galves-Löcherbach dynamics

We define from the initial conditions $\{X_i\}$ the state variables at time 1 $\{Y_i\}$ according to the following dynamics:

$$Y_i = X_i \mathbb{1}_{\{U_i > \sigma_i(X_i)\}} + A_i, \quad \forall i = 1, \ldots, K, \tag{4}$$

with arrivals processes

$$A_i = \sum_{j \neq i} \mu_{i,j} \mathbb{1}_{\{U_j < \sigma_j(X_j)\}}, \quad \forall i = 1, \ldots, K, \tag{5}$$

with i.i.d. r.v.s $\{U_i\}$ uniformly distributed on $[0, 1]$ and independent from $\{X_i\}$ and activation probabilities $\{\sigma_i(0), \sigma_i(1), \ldots\}_{i \in \{1,\ldots,K\}}$ such that $\sigma_i(0) = 0$ and $0 < \sigma_i(1) \leq \sigma_i(2) \leq \cdots \leq 1$ for all $i$; The continuation of the dynamics is defined by induction.

## The associated RMF model

The integer-valued state variables at time 1, denoted $\{Y^M_{n,i}\}$, are given by the equations

$$Y^M_{n,i} = X^M_{n,i}\, \mathbb{1}_{\{U_{n,i} > \sigma_i(X^M_{n,i})\}} + A^M_{n,i}, \tag{6}$$

where $\sigma_i$ are the activation probabilities, where

$$A^M_{n,i} = \sum_{m \neq n} \sum_{j \neq i} \mu_{i,j}\, \mathbb{1}_{\{U_{m,j} < \sigma_i(X^M_{m,j})\}}\, \mathbb{1}_{\{R^M_{m,j,i} = n\}} \tag{7}$$

represents the aggregations to node $i$ in replica $n$ and where $\{R^M_{m,j,i}\}$ are routing variables independent from $\{X^M_{n,i}\}$ and $\{U_{n,i}\}$, uniformly distributed on $\{1, \ldots, M\} \setminus \{m\}$ for all $i, j \in \{1, \ldots, K\}$ and $m \in \{1, \ldots, M\}$.

### Objective

Intuitively at the limit:

- $A^M_{n,i}$ sum of rare events converges in distribution to some kind of Poisson law.
- The probability that two nodes in two given replicas interact is in $\frac{1}{M}$ : asymptotic independence between replicas

## Pairwise asymptotic independence

We consider $M \in \mathbb{N}$, a set of integer-valued r.v.s $Z = \{Z^M_{n,i}\}_{1 \leq n \leq M, 1 \leq i \leq K}$ such that for all fixed $M$, the r.v.s $Z^M_{n,i}$ are *exchangeable* in $n$. We say that $Z^M_{n,i}$ are pairwise asymptotically independent, or that $Z$ verifies PAI($Z$), if there exist integer-valued r.v.s $(\tilde{Z}_i)_{i \in \{1,\ldots,K\}}$ such that $\forall (n,i) \neq (m,j), \forall u, v \in [0,1]$,

$$\lim_{N \to \infty} \mathbf{E}[u^{Z^N_{n,i}} v^{Z^N_{m,j}}] = \mathbf{E}[u^{\tilde{Z}_i}]\, \mathbf{E}[v^{\tilde{Z}_j}]. \quad (8)$$

**The main result (F. Baccelli, M.D. and T. Taillefumier)**

Let $M \in \mathbb{N}$, let $X = \{X_{n,i}^M\}_{n\in\{1,\dots,M\},i\in\{1,\dots,K\}}$ be an array of integer-valued r.v.s. Suppose that PAI($X$) holds. Then we have PAI($Y$), where $Y$ is defined by the RMF dynamics (6). Moreover, the arrivals to a given node $A_{n,i}^M$ converge in distribution to a compound Poisson distributed random variable.

## Idea of proof

- Write $Y^M_{n,i} = \hat{X}^M_{n,i} + A^M_{n,i}$ (dynamics at time 1=autonomous evolution+arrivals)
- Show that PAI($X$) implies PAI($\hat{X}$) (direct computation)
- Prove the convergence of the PGF of $A^M_{n,i}$ to that of a compound Poisson distributed r.v. using the *triangular law of large numbers*
- Show that PAI($\hat{X}$) and PAI($A$) imply PAI($Y$).

## Triangular law of large numbers (TLLN)

Given $M \in \mathbb{N}$, given an array of integer-valued random variables $Z = \{Z_n^M\}_{n \in \{1,\dots,M\}}$ such that for all fixed $M$, the random variables $Z_n^M$ are exchangeable in $n$, we say that $Z$ verifies TLLN($Z$), if there exists an integer-valued random variable $\tilde{Z}$ such that for all functions $f : \mathbb{N} \to \mathbb{R}$ with compact support, we have the following limit in $L^2$:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(Z_n^N) = \mathbf{E}[f(\tilde{Z})].$$

## Link between PAI and TLLN

Let $M \in \mathbb{N}$, let $Z = \{Z_{n,i}^M\}_{n \in \{1,\dots,M\}, i \in \{1,\dots,K\}}$ be an array of integer valued random variables verifying PAI($Z$). Then, for all $i$, $Z_i = \{Z_{n,i}^M\}_{n \in \{1,\dots,M\}}$ satisfies TLLN($Z_i$).

## Explicit formula for discrete GL neural networks

Supposing that PAI($X$) holds, we have propagation of chaos in this system, and the limit distributions of arrivals at the different nodes are characterized by, for $i \in \{1, \ldots, K\}$ and $z \in [0, 1]$,

$$\mathbf{E}\left[z^{\tilde{A}_i}\right] = e^{\theta_i \sum_{j \neq i}(z^{\mu_{i,j}} - 1)} = \prod_{j \neq i} e^{\theta_i (z^{\mu_{i,j}} - 1)},$$

where $\theta_i = \mathbf{E}\left[\sigma_i(\tilde{X}_i)\right]$.

# Link between discrete and continuous RMFs

## Markov description of FIAPs

The RMF FIAP GL model forms a discrete-time discrete space Markov chain with transition given by:

$$\begin{array}{c} (X^M_{1,1}, \dots, X^M_{1,K}, X^M_{2,1}, \dots, X^M_{M,K}) \\ \downarrow \\ (X^M_{1,1}, \dots, r_{i_1}, \dots, r_{i_L}, \dots, X^M_{M,K}) \\ \downarrow \\ \left(X^M_{1,1}, \dots, X^M_{n_{i_1},1} + \mu_{i_1,1}, \dots, X^M_{n_{i_k},j} + \mu_{i_k,j}, \dots, X^M_{M,K}\right). \end{array} \tag{9}$$

In the first half-step, $L$ variables are reset to their base value $r_{i_l}$ for $1 \leq l \leq L$ for some $L$ in $\{0, \dots, MK\}$. In the second half-step, for each variable $X^M_{m_l,i_l}$ reset in the first half-step, for each $j \neq i_l$, one variable $X^M_{n_l,j}$ (with $n_l \neq m_l$) chosen uniformly at random is incremented by $\mu_{i_l,j}$.

## Approximation of the continuous RMF

We consider a discrete time Markov chain with transitions given by (9) with the associate transition probability, where $L \in \{1, \ldots, KM\}$ and $\mathcal{L} = \{(m_1, i_1), \ldots, (m_L, i_L)\}$, but where the reset probability $p^M_{n,j}(\delta)$ is equal to the probability that neuron $(n, j)$ in the continuous-time RMF GL model spikes at least once in a unit of time of length $\delta$ given its state at the beginning of the unit of time. Namely, we set $\sigma_\delta(k) = 1 - e^{\delta\sigma(k)}$, where $\sigma(k)$ is the spiking rate of a neuron in the RMF GL model in state $k$.

### Reverse construction

Given RMF FIAP dynamics of the type defined above for all $\delta > 0$, since for all $k$, $\sigma_\delta(k) = \delta\sigma(k) + o(\delta)$, we can reconstruct the infinitesimal generator of the continuous-time dynamics by considering the transition operator $\frac{1}{\delta}(P_\delta - Id)$, where $Id$ is the identity operator and $P_\delta$ is the transition operator of the RMF FIAP dynamics with time steps of length $\delta$.

## Summarizing diagram

$$
\begin{array}{ccc}
\text{infinite-replica discrete GL} & \underset{\longleftarrow}{\overset{\delta\to 0}{\longrightarrow}} & \text{infinite-replica continuous GL} \\
{\scriptstyle M\to\infty}\uparrow & & {\scriptstyle M\to\infty}\uparrow \\
M\text{-replica discrete GL with } \delta\text{-time step} & \underset{\longleftarrow}{\overset{\delta\to 0}{\longrightarrow}} & M\text{-replica continuous GL}
\end{array}
$$

# Conclusion





## Conclusion

- RMF model: a mean field type hypothesis that preserves the geometry of the initial network.
- Main results : proof of Poisson hypothesis in both discrete and continuous time on large classes of processes.
- RMF markovian models in discrete and continuous time can be otbtained from one another.

## Open questions

- Strong Poisson hypothesis : commuting time and replica limits
- Relaxing hypotheses on initial conditions and/or weights