Law of large numbers and limit central theorem for auto-inhibited Hawkes processes

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Definition

Let $h : (0, +\infty) \to \mathbb{R}$ a signed measurable function.
A Hawkes process N^h is a self-influencing point process whose intensity is given at each time $t \geq 0$ by:

$$\Lambda^h(t) = \Phi\left(\int_{(-\infty,t)} h(t-u)N^h(du)\right) = \Phi\left(\sum_{i\geq 1} h(t-U_i)\right)$$

where $\Phi : \mathbb{R} \to \mathbb{R}^+$, and U_i are the jumps of N^h.

Definition

Linear process

If Φ is linear or affine ($\lambda \geq 0$), and h is positive:

$$\Lambda^h(t) = \lambda + \int_{(-\infty,t)} h(t-u)N^h(du).$$

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Our case

$\Phi(x) = \max(0, \lambda + x)$ and signed h:

$$\Lambda^h(t) = \left(\lambda + \int_{(-\infty,t)} h(t-u)N^h(du)\right)^+.$$

Literature

We study the number of jumps on an interval $[0, t]$:

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For linear processes, $h \geq 0$ s.t. $\| h \|_1 < 1$, $\lambda \geq 0$:

$$\frac{N^h_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\lambda}{1 - \| h \|_1}$$

$$\frac{1}{\sqrt{t}}\left(N^h_t - \frac{\lambda}{1-||h||_1}t\right) \xrightarrow[t\to\infty]{law} \mathcal{N}^h(0,\sigma^2) \quad \text{with } \sigma^2 = \frac{\lambda}{(1 - \| h \|_1)^3}$$

Hypothesis

- $\| h^+ \|_1 < 1$
- Empty initial condition: $N^h(]-\infty, 0]) = 0$
- h has a compact support, included in $[0, L(h)]$

Cumulative process

Definition

Let $(\tau_i, W_i)_i$ i.i.d. couples of random variable.
Let M_t the renewal process associated with $(\tau_i)_i$:

$$M_t = \sup_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} \tau_i \leq t \right\}.$$

The *cumulative process* associated with $(\tau_i, W_i)_i$ is

$$Z_t = \sum_{i=1}^{M_t} W_i.$$

LLN and LCT

Law of Large Numbers for cumulative process

If $\mathbb{E}[\tau_1] < \infty$ and $\mathbb{E}[|W_1|] < \infty$, then

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$$

Limit Central Theorem for cumulative process

If $Var[\tau_1] < \infty$ and $Var[|W_1|] < \infty$, then

$$\frac{1}{\sqrt{t}}\left(Z_t - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}t\right) \xrightarrow[t\to\infty]{law} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = Var\left(W_1 - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right)$.

Link between Hawkes process and cumulative process

Hawkes process is *almost* a cumulative process.

Intensity

$\Lambda^h(t) = \left(\lambda + \sum_{i\geq 1} h(t-U_i)\right)^+$, where U_i are the jumps of N^h.
If $t > U_i + L(h)$ for each $U_i < t$, then $\Lambda^h(t) = \lambda$.

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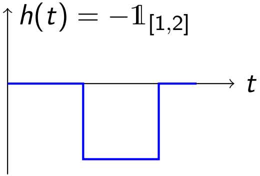
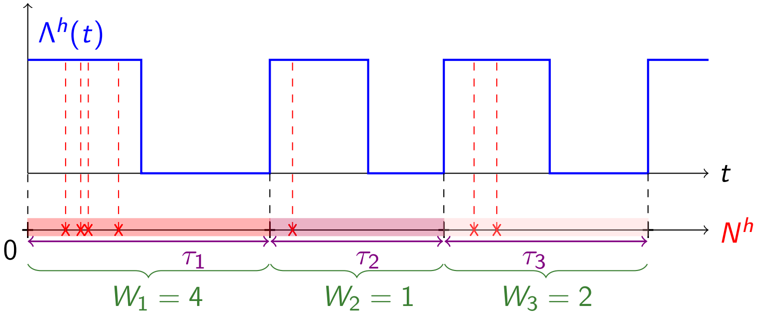
We can define

$\tau_1 = \inf\{t > U_1^1, N^h((t - L(h), t]) = 0\}$,
$W_1 = N^h([0, \tau_1])$, ...

Definition of τ and W





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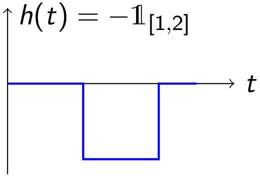
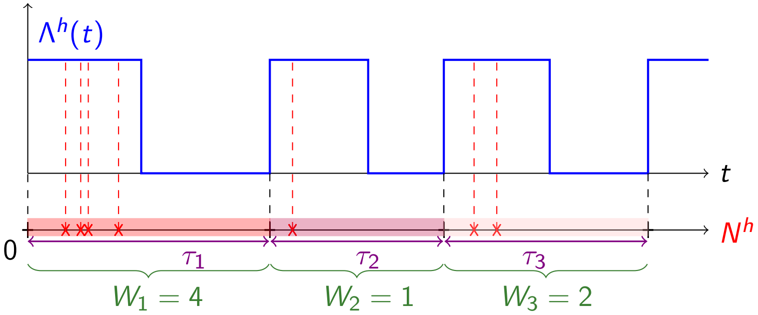




$\tau_1 = \inf\{t > U_1^1, N^h((t - L(h), t]) = 0\}$

$W_1 = N^h([0, \tau_1])$.

$N^h(t) = \sum_{i=1}^{M_t^h} W_i + R_t$

Propositions

Law of large numbers for Hawkes process

Let h be a signed function, with a support includes in $[0, L(h)]$. Then we have:

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}.$$

Limit Central Theorem for Hawkes process

Let h be a signed function, with a support includes in $[0, L(h)]$. Then we have:

$$\frac{1}{\sqrt{t}}\left(N_t^h - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}t\right) \xrightarrow[t\to\infty]{law} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = Var\left(W_1 - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right)$.

Sketch of the proof

- $\dfrac{N_t^h}{t} = \dfrac{1}{t}\sum_{i=1}^{M_t^h} W_i + \dfrac{R_t}{t}$.
- We prove $Var(\tau_1) < \infty$ and $Var(W_1) < \infty$.
- We apply the theorem (LLN or LCT) for cumulative process $\dfrac{1}{t}\sum_{i=1}^{M_t^h} W_i$. We have:

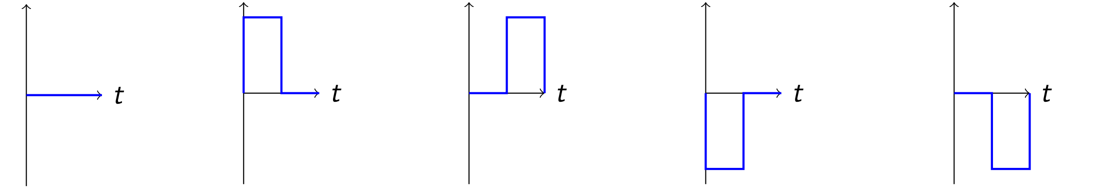
$$\frac{1}{t}\sum_{i=1}^{M_t^h} W_i \xrightarrow[t\to\infty]{a.s} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}, \quad \text{and} \quad \frac{1}{\sqrt{t}}\left(\sum_{i=1}^{M_t^h} W_i - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}t\right) \xrightarrow[t\to\infty]{law} \mathcal{N}(0,\sigma^2)$$

- We prove $\dfrac{R_t}{\sqrt{t}} \xrightarrow[t\to\infty]{\mathbb{P}} 0$.
- Then

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}, \quad \text{and} \ \frac{1}{\sqrt{t}}\left(N_t^h - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}t\right) \xrightarrow[t\to\infty]{law} \mathcal{N}(0,\sigma^2)$$

Examples and conclusion

Let $\lambda = 1$ and let:



$h_0(t) = 0$ $\quad$ $h_1(t) = \mathbb{1}_{[0,0.5]}$ $\quad$ $h_2(t) = \mathbb{1}_{[0.5,1]}$ $\quad$ $h_3(t) = -\mathbb{1}_{[0,0.5]}$ $\quad$ $h_4(t) = -\mathbb{1}_{[0.5,1]}$

Poisson process:

$$\frac{N_t^{h_0}}{t} \xrightarrow[t\to\infty]{a.s.} \lambda = 1$$

$$\| h_1^+ \|_1 = \| h_2^+ \|_1 = \frac{1}{2}$$

$$\frac{N_t^{h_1}}{t} \xrightarrow[t\to\infty]{a.s.} 2 \text{ and } \frac{N_t^{h_2}}{t} \xrightarrow[t\to\infty]{a.s.} 2$$

$$\lim = \frac{\lambda}{1 - \| h_1^+ \|_1} = 2$$

$$\frac{N_t^{h_3}}{t} \xrightarrow[t\to\infty]{a.s.} \frac{2}{3}$$

$$\lim = \frac{\lambda}{1 + \lambda L(h_3)}$$

$$\frac{N_t^{h_4}}{t} \xrightarrow[t\to\infty]{a.s.} \frac{1.5}{1.5 + e^{-0.5}} \simeq 0.71$$

$$\lim = \frac{\lambda(1+\lambda r)}{\lambda L(h_4) + e^{-\lambda r} + \lambda r}$$

with $r = 0.5$

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