Multivariate Hawkes processes on inhomogeneous random graphs

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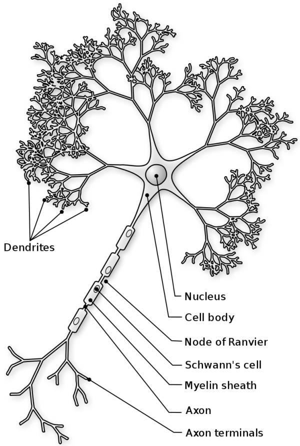
Sommaire

$\sim 86.10^9$ neurons in human brain

Reception of information

- Permeable membrane
- Presynaptic neuron $\Rightarrow$ release of neurotransmitters

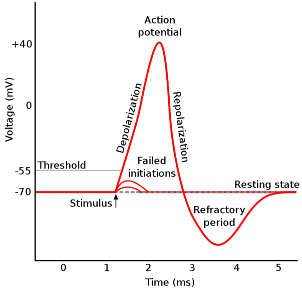


Integration of information

- All-or-none reaction : stimulus > threshold $\Rightarrow$ **spike** emission

Transfer of information

- Propagation to postsynaptic neurons
- Cell-to-cell communication



Interaction

- Spatial organization, e.g. cortical columns [Bosking et al., 1997, Mountcastle, 1997]

Modelling of N neurons in interaction on a graph

- The neuron i is located in $x_i \in I \subset \mathbb{R}^d$
- $\left(Z_i^{(N)}(t)\right)_{t\geq 0}$ counting process :
  $Z_i^{(N)}(t) = \{\text{number of spikes of the neuron } i \text{ on } [0,t]\}$
- conditional intensity at time t :

$$\lambda_i^{(N)}(t) = f\left(u_0(t,x_i) + \frac{1}{N}\sum_{j=1}^{N} w_{ij}^{(N)} \int_0^{t-} h(t-s)dZ_j^{(N)}(s)\right). \tag{1}$$

$\Rightarrow \left(Z_1^{(N)}(t), \cdots, Z_N^{(N)}(t)\right)_{t>0}$ multivariate Hawkes process.

Firing intensity

$$\lambda_i^{(N)}(t) = f\left(u_0\left(t-, x_i\right) + \frac{1}{N}\sum_{j=1}^{N} w_{ij}^{(N)} \int_{]0,t[} h\left(t-s\right) dZ_j^{(N)}(s)\right)$$

- $f \ : \ \mathbb{R} \longrightarrow \mathbb{R}_+$: synaptic integration (ex : linear, sigmoid...)
- $u_0 \ : \ \mathbb{R}_+ \times I \longrightarrow \mathbb{R}$: spontaneous activity of the neuron (ex : linear, gaussian...)
- $h \ : \ \mathbb{R}_+ \longrightarrow \mathbb{R}$: memory function which models how a past jump of the system affects the present intensity (ex : decreasing exponential $h(t) = e^{-\alpha t}$, compact support $h(t) = 1_{0 \leq t \leq \theta}$)
- $w_{ij}^{(N)}$: interaction between the neurons i and j

Graph of interaction

Interaction between neurons i and j : $w_{ij}^{(N)} = \kappa_i^{(N)} \xi_{ij}^{(N)}$ where

- $\kappa_i^{(N)} \geq 0$ dilution parameter so that the interaction term remains of order 1 as $N \to \infty$
- $\forall i, j$, $\xi_{ij}^{(N)} \in \{0,1\} \sim \mathcal{B}\left(W_N\left(x_i, x_j\right)\right)$ where $W_N \; : \; I \times I \longrightarrow [0,1]$ kernel of the microscopic structure

Definition

Graph of interaction : $\mathcal{G}^{(N)} = \left(\{1, \cdots, N\}, \left(\xi_{ij}^{(N)}\right)_{1 \leq i,j \leq N} \right)$

Annealed graph :

$\mathcal{G}_N^{(1)} = \left(\{1, \cdots, N\}, \text{edge } j \to i \text{ with weight } \kappa_i^{(N)} W_N(x_i, x_j) \right)$

Some dense graphs, $I = [0, 1], x_i = \frac{i}{N}, \kappa_i^{(N)} = 1, i = 1 \dots N$

Erdös-Rényi graph

$W_N(x, y) = \rho_N$ with $\rho_N \to \rho > 0$

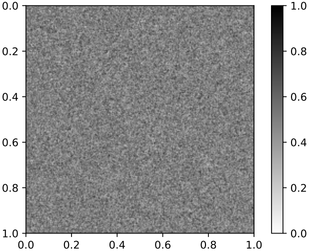


Figure – $\mathcal{G}^{(N)}$ with $\rho_N = 0.5$, $N = 1000$

P-nearest neighbor

$W_N(x, y) = 1_{\min(|x-y|, 1-|x-y|)<r}$

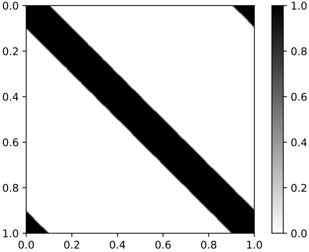


Figure – $\mathcal{G}^{(N)}$ with $r = 0.1$, $N = 500$

Inhomogeneous graph, $I = [0,1], x_i = \frac{i}{N}, \kappa_i^{(N)} = 1, i = 1 \dots N$

Expected Degree Distribution (EDD)
$W_N(x,y) = g(x)k(y)$

Here : $W_N(x,y) = xy$

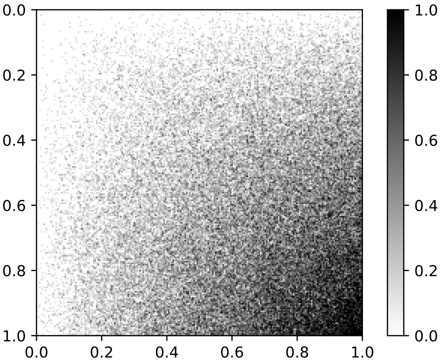


Figure – $\mathcal{G}^{(N)}$ with $N = 500$

$(\pi_i(ds, dz))_{1 \leq i \leq N}$: i.i.d. Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dsdz$. For all $t \geq 0, i \in [\![1, N]\!]$:

$$Z_i^{(N)}(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda_i^{(N)}(s)\}} \pi_i(ds, dz).$$

Hypotheses

f Lipschitz continuous (L_f), h locally integrable, u_0 time continuous, Lipschitz continuous in space and uniformly bounded.

Proposition [Delattre et al., 2016, Chevallier et al., 2019]

For a fixed realisation of $(\pi_i)_{1 \leq i \leq N}$, there exists a pathwise unique multivariate Hawkes process $\left(Z_1^{(N)}(t), \cdots, Z_N^{(N)}(t)\right)_{t>0}$ such that $\left(\sup_{1 \leq i \leq N} \mathbb{E}[Z_i^{(N)}(t)]\right)_{t \geq 0}$ is locally bounded.

Sommaire

$$\lambda_i^{(N)}(t) = f\left(u_0(t,x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^{N} \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right) \xrightarrow[N\to\infty]{} ?$$

About the positions

- $\nu^{(N)} := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(dx) \xrightarrow[N\to\infty]{} \nu$, a macroscopical distribution

Scenario (1) : $I = [0,1]$ and $x_i = \frac{i}{N} \Rightarrow \nu(dx) = dx$
Scenario (2) : (x_i) i.i.d. of distribution ν

About the graph

$\mathcal{G}^{(N)} \xrightarrow[N\to\infty]{} ?$

Macroscopic interaction graphon $W : I^2 \to \mathbb{R}_+$

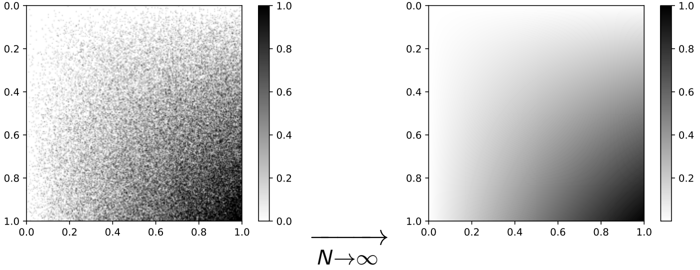


Figure – EDD case - $\mathcal{G}^{(N)}$ with $N = 500$ and the graphon $W(x,y) = xy$

- Graph $\mathcal{G}^{(N)}$ $\Rightarrow$ Connectivity matrix $\Rightarrow$ step-function $W^{\mathcal{G}^{(N)}}$ (graphon) [Lovász, 2012]
- Cut-distance between W_1 and W_2 graphons :
  $d_{\square,\nu}(W_1, W_2) = \sup_{S,T \subset I} \left| \int_{S \times T} (W_1 - W_2)(x,y)\, \nu(dx)\nu(dy) \right|$

Heuristics of the limiting intensity

$$\lambda_i^{(N)}(t) = f\left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^{N} \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right)$$

$$\text{"} \xrightarrow[N\to\infty]{} \text{"}$$

$$\lambda(t, x) = f\left(u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s)\lambda(s, y) ds \nu(dy) \right) \qquad (2)$$

Proposition - Let $T > 0$. Assume

- $D(x) := \int_I W(x, y)\nu(dy)$, $\sup_{x \in I} D(x) < \infty$
- $\int_I |W(x, y) - W(x', y)|\nu(dy) \leq C\|x - x'\|^{\iota}$, $\iota \in (0, 1]$

There exists a unique solution λ of (2) continuous and bounded on $[0, T] \times I$.

Coupling [Delattre et al., 2016, Chevallier et al., 2019]

For $t \in [0, T], i \in [\![1, N]\!]$ and the same (π_i) :

- $Z_i^{(N)}(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda_i^{(N)}(s)\}} \pi_i(ds, dz)$ with

$$\lambda_i^{(N)}(t) = f\left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^{N} \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s)\right)$$

- $\overline{Z}_i(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda(s, x_i)\}} \pi_i(ds, dz)$ with

$$\lambda(t, x) = f\left(u_0(t, x) + \int_{\mathbb{R}^d} W(x, y) \int_0^{t-} h(t-s) \lambda(s, y) ds \nu(dy)\right).$$

Our hypotheses

- Control of the dilution of the graph $\mathcal{G}^{(N)}$
- Control of indegrees :
  $\sup_{i\in[\![1,N]\!]} \frac{1}{N}\sum_{j=1}^N \kappa_i^{(N)} W_N(x_i,x_j) \leq C_W$
- $d_{\square,\nu}\left(W^{\mathcal{G}_N^{(1)}}, W\right) \xrightarrow[N\to\infty]{} 0$
- Control of outdegrees :
  $\sup_{j\in[\![1,N]\!]} \frac{1}{N}\sum_{i=1}^N \kappa_i^{(N)} W_N(x_i,x_j) \leq C_W$

Theorem [A.-N. 2021] - Let $T > 0$, for $\mathbb{P}$-almost realisations of the connectivity sequence $\left(\xi^{(N)}\right)_{N\geq1}$ and positions $\left(\underline{x}_N\right)_{N\geq1}$:

$$\frac{1}{N}\sum_{i=1}^N \mathbf{E}\left[\sup_{t\in[0,T]}\left|Z_i^{(N)}(t) - \overline{Z}_i(t)\right|\right] \xrightarrow[N\to\infty]{} 0.$$

Hypotheses

- $\|W^{\mathcal{G}_N^{(1)}} - W\|_{\infty\to\infty,\nu} \xrightarrow[N\to\infty]{} 0$ with

$$\|W\|_{\infty\to\infty,\nu} := \sup_{\|g\|_\infty \leq 1} \sup_{x\in I} \left| \int_I W(x,y) g(y) \nu(dy) \right| .$$

Theorem [A.-N. 2021]

Let $T > 0$, for $\mathbb{P}$-almost realisations of the connectivity sequence $\left(\xi^{(N)}\right)_{N\geq 1}$ and positions $(\underline{x}_N)_{N\geq 1}$:

$$\max_{1\leq i\leq N} \mathbf{E}\left[\sup_{t\in[0,T]} \left| Z_i^{(N)}(t) - \overline{Z}_i(t) \right| \right] \xrightarrow[N\to\infty]{} 0.$$

Empirical measure

Notation

Define the measures on $S := \mathbb{D}([0, T], \mathbb{N}) \times I$:

- $\mu_N(d\eta, dx) := \frac{1}{N} \sum_{i=1}^{N} \delta_{\left(Z_i^{(N)}([0,T]), x_i^{(N)}\right)}(d\eta, dx)$
- $\mu_\infty(d\eta, dx) := P_{[0,T],\infty}(d\eta|x)\, \nu(dx)$,

for $P_{[0,T],\infty}(\cdot|x)$ distribution of an inhomogeneous Poisson point process with intensity $(\lambda(t, x))_{0 \le t \le T}$.

Empirical measure

Proposition [A.-N. 2021]

For $\mathbb{P}$-almost realisations of the connectivity sequence $\left(\xi^{(N)}\right)_{N\geq 1}$ and positions $(\underline{x}_N)_{N\geq 1}$,

$$\mathrm{E}\left[d_{BL}\left(\mu_N, \mu_\infty\right)\right] \xrightarrow[N\to\infty]{} 0$$

where

$$d_{BL}(\mu, \nu) := \sup_{g, \|g\|_{BL}\leq 1} \left| \int_S g\left(d\mu - d\nu\right) \right|$$

$$\|g\|_{BL} := \|g\|_{Lip} + \|g\|_{S,\infty}$$

Spatial profile, with $I = [0,1]$ and $x_i = \frac{1}{N}$

Notation - Define the functions

- $U_N(t,x) := \sum_{i=1}^N U_{i,N}(t) 1_{x \in (\frac{i-1}{N}, \frac{i}{N}]}$, where

$$U_{i,N}(t) := u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s)$$

- $u(t,x) := u_0(t,x) + \int_I W(x,y) \int_0^t h(t-s) \lambda(s,y) ds\, \nu(dy)$

When $h(t) = e^{-\alpha t}$, u is the solution of the scalar neural field equation [Amari, 1977, Wilson - Cowan, 1972, Chevallier et al., 2019]

$$\partial_t u(t,x) = -\alpha u(t,x) + \int_I W(x,y) f(u(t,y)) \nu(dy).$$

Spatial profile, with $I = [0, 1]$ and $x_i = \frac{1}{N}$

Proposition [A.-N. 2021]

For $\mathbb{P}$-almost realisations of the connectivity sequence $\left(\xi^{(N)}\right)_{N\geq 1}$ and positions $(\underline{x}_N)_{N\geq 1}$,

$$\mathrm{E}\left[\int_0^T \int_0^1 |U_N(t,x) - u(t,x)|\, dx \ dt\right] \xrightarrow[N\to\infty]{} 0.$$

Sommaire

Linear case : $f = Id$

$$\lambda(t,x) = u_0(t,x) + \int_I W(x,y) \int_0^t h(t-s)\lambda(s,y)ds \, \nu(dy)$$

Without spatial interaction [Delattre et al., 2016]

$$\lambda(t) = u_0 + \int_0^t h(t-s)\lambda(s)ds$$

Phase transition

- Subcritical case ($\|h\|_1 < 1$) : $\lambda(t) \xrightarrow[t\to\infty]{} \dfrac{u_0}{1-\|h\|_1}$
- Supercritical case ($\|h\|_1 > 1$) : $\lambda(t) \sim \alpha e^{\beta t} \to \infty$ for some $\alpha, \beta > 0$

$$
\begin{aligned}
\lambda(t,x) &= u_0(t,x) + \int_I W(x,y) \int_0^t h(t-s)\lambda(s,y)ds \ \nu(dy) \\
&= u_0(t,x) + \int_0^t h(t-s) T_W \lambda(s,\cdot)(x) ds
\end{aligned}
$$

Integral operator T_W

$$
\begin{array}{rccl}
T_W : & L^\infty(I) & \longrightarrow & L^\infty(I) \\
& g & \longmapsto & \left(T_W g : x \longmapsto \int_I W(x,y) g(y) \nu(dy)\right).
\end{array}
$$

Spectral radius

$$
r_\infty := r_\infty(T_W) = \sup_{\sigma \in Sp(T_W)} |\sigma| = \lim_{n\to\infty} \| T_W^n \|^{\frac{1}{n}}.
$$

Subcritical case : $\|h\|_1 r_\infty < 1$

Theorem [A.-N. 2021]

In the subcritical case $\|h\|_1 r_\infty < 1$, if $\sup_{x\in I} |u_0(t,x) - u(x)| \xrightarrow[t\to\infty]{} 0$, then for any $x \in I$,

$$\lambda(t,x) \xrightarrow[t\to\infty]{} \ell(x)$$

where ℓ is the unique continuous and bounded function solving

$$\ell(x) = u(x) + \|h\|_1 \int_I W(x,y)\ell(y)\nu(dy).$$

Proposition [A.-N. 2021]

If u_0 is constant, ℓ is uniform if and only if the indegree is uniform ($\int_I W(x,y)\nu(dy) = D$ for every $x \in I$). In such case, $r_\infty = D$.

Subcritical case - Erdös-Rényi graph

$$W_N(x,y) = \rho$$
$$N = 1000$$

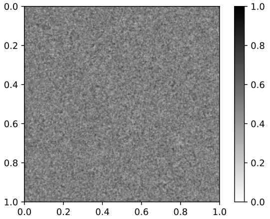


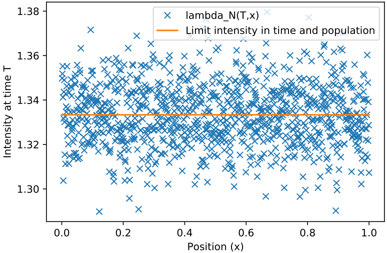
Figure – $\mathcal{G}^{(N)}$

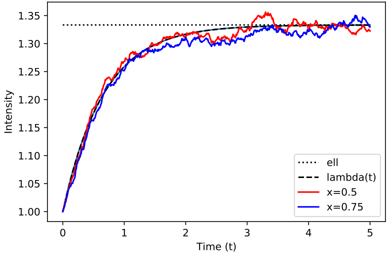
Subcritical condition : $\|h\|_1 \rho < 1$

$$\ell(x) = \frac{u(x)(1 - \|h\|_1\rho) + \|u\|_{I,1}\|h\|_1\rho}{1 - \|h\|_1\rho}$$

Subcritical case - Erdös-Rényi graph with homogeneous u_0

$$r_\infty = \rho = 0.5, \quad h(t) = e^{-2t}, \quad u_0(t,x) = 1$$

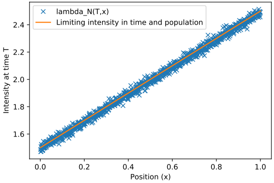


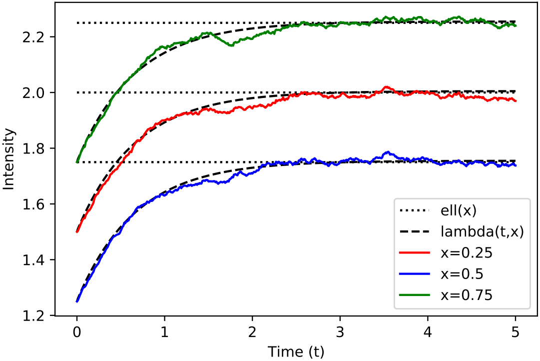


$$\lambda(t) = \frac{4}{3} - \frac{1}{3} e^{-\frac{3}{2}t}, \quad \ell = \frac{4}{3}$$

Subcritical case - Erdös-Rényi graph with inhomogeneous u_0

$$r_\infty = \rho = 0.5$$
$$h(t) = e^{-2t}$$
$$u_0(t,x) = x + 1$$
$$\ell(x) = x + \frac{1}{2}$$





Subcritical case - P-nearest neighbor

$$W_N(x,y) = 1_{\min(|x-y|,1-|x-y|)<r}, \quad r_\infty = 2r$$
$$h(t) = e^{-2t}, \quad u_0(t,x) = 1, \quad r = 0.1$$

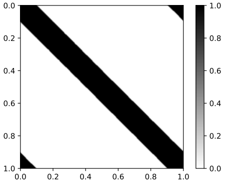
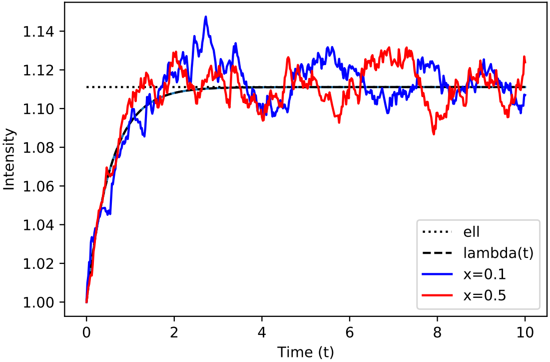


Figure – $\mathcal{G}^{(N)}$



$N = 500$
$\lambda(t) = \frac{10}{9} - \frac{1}{9} e^{-\frac{9}{5}t}$

Subcritical case - EDD

f,g densities on I, bounded

$$W_N(x,y) = f(x)g(y), \quad D(x) = f(x), \quad r_\infty = \langle f, g\rangle$$

When $\|h\|_1 \langle f, g\rangle < 1$:

$$\lambda(t,x) \xrightarrow[t\to\infty]{} \ell(x) = u(x) + \|h\|_1 \frac{f(x)\langle u, g\rangle}{1 - \|h\|_1\langle f, g\rangle}.$$

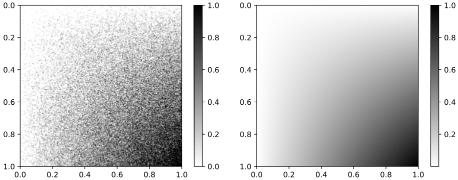
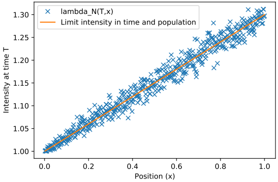
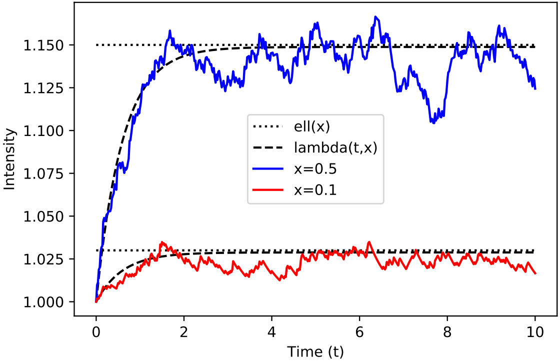


Figure – $\mathcal{G}^{(N)}$ with $N = 500$ and the graphon $W(x,y) = xy$

Subcritical case - EDD $W(x, y) = xy$

$$h(t) = e^{-2t}$$
$$u_0(t, x) = 1$$
$$\ell(x) = 1 + \frac{3}{10}x$$



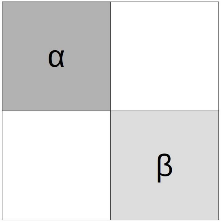


Supercritical case : $\|h\|_1 r_\infty > 1$

$$\|h\|_1 r_\infty > 1 \Rightarrow \lambda(t,x) \xrightarrow[t\to\infty]{} \infty \text{ ?}$$

Example

W with 2 disconnected mean-field components, $\alpha > \beta$, $r_\infty = \alpha/2$:
A population can be in the subcritical case and the other in the supercritical case.



(critical parameters : $\alpha_c = \frac{2}{\|h\|_1}$ and $\beta_c = \frac{2}{\|h\|_1}$)

Supercritical case : $\|h\|_1 r_\infty > 1$

Hypotheses

- $\sup_x \int_I W(x,y)^2 \nu(dy) =: C_{W_2} < \infty$,
- $\forall (x,y) \in I^2, \quad W(x,y) = W(y,x)$
- there exists k such that $W^{(k)} > 0$ where $W^{(k)}(x,y) := \int_{I\times\dots\times I} W(x,x_1)\cdots W(x_{k-1},y)dx_1\cdots dx_{k-1}$.

Proposition [A.-N. 2021]

$$\int_I \lambda(t,x)^2 \nu(dx) \xrightarrow[t\to\infty]{} \infty$$

Thanks !

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